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POSSIBLE PARITY ASSIGNMENTS FOR THE STRANGE PARTICLES

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Abstract: A general method is described which enables parities and coupling constants to be determined from angular distributions for strange particle processes. In particular from associated production of Λ and Σ particles by pions the relative parity of K^+ to K^0 seems quite conclusively to be odd, while the data are fitted best by K^+ having odd parity with respect to both the (ΛN) and the (ΣN) systems. Some inequalities between combinations of various coupling constants are also derived.

1. Introduction

There have been various attempts [1]) to determine the relative parities of the strange particles, though none have given any definite result. We present here a new method for the determination of these parities, together with some tentative parity assignments which follow from present experimental data. Our method also leads to an inequality between certain coupling constants, and will give numerical values for some of these coupling constants when suitable data are available.

2. Method

The method we use depends on a conjecture concerning the analyticity of scattering and production amplitudes (for strong interactions), regarded as functions of momentum transfer. This conjecture has already been used to obtain the pion-nucleon interaction coupling constant [2]). We cannot at present give a proof of this conjecture, but we shall give now a heuristic argument for it.

We shall first consider associated production by pions. The analysis of this process gives a lot of information about the various parity assignments which we are considering. We shall later discuss other processes which may be used, when the relevant experimental data are forthcoming, to check our parity assignments.

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Let us consider, then, the process

$$\pi+\mathrm{N} \to \mathrm{K}+\mathrm{Y} \tag{1}$$

where Y is Λ or Σ. Let k, q, p, p' be the pion, K-meson, nucleon and Y-particle momenta. We neglect the spin and isobaric spin variables of the participating particles in this section since doing so does not affect the analyticity of the production amplitude, provided we take into account correctly all selection rules. The production matrix element may be taken to be a function $M(W, \Delta^2)$ of the invariants $W = -(k+p)^2$, $\Delta^2 = (k-q)^2$.

We consider also the annihilation process

$$\mathrm{N}+\overline{\mathrm{Y}} \to \pi+\mathrm{K}. \tag{2}$$

If k, q, p, p' are the pion, K-meson, nucleon and anti-Y momenta then the annihilation matrix element is a function $N(W', \Delta'^2)$ of the invariants $W' = -(p+p')^2$, $\Delta'^2 = (p-k)^2$. Since the transformation $p' \to -p'$, $k \to -k$ transforms process (1) into process (2) and vice-versa (crossing symmetry), one has

$$N(W', \Delta'^2) = M(-\Delta'^2, -W'). \tag{3}$$

From the analyticity behaviour of $N(W', \Delta'^2)$ in W' for fixed Δ'^2 (dispersion relation behaviour), we obtain the analyticity of $M(W, \Delta^2)$ in Δ^2 for fixed W. We denote by θ the angle between the incoming pion and the produced K-meson, in the centre of mass system. Then by standard methods we may show that the region of analyticity of M is the cut $\cos\theta$-plane, with simple poles at $\cos\theta = (2k_0q_0-\mu^2)/2kq = \alpha_0$ and $\cos\theta = (-Y'^2+M^2+\kappa^2-2q_0p_0)/2kq = \beta_0$, arising from intermediate states composed of a single K-meson and a single Y-particle respectively, and Y′ is Λ or Σ or both, as allowed by charge conservation. Y', M, κ are the masses of the Y′-particle, nucleon, and K-meson. The branch points are from $\cos\theta = (k_0q_0+\kappa\mu)/kq = \alpha_1$ to $+\infty$, and from $\cos\theta = (-(Y'+\mu)^2+M^2+\kappa^2-2q_0p_0)/2kq = \beta_1$ to $-\infty$, arising from intermediate states with at least one pion and one K-meson, or one pion and one Y′-particle.

Then we expect to be able to write for the production amplitude

$$M(W, \cos\theta) = \frac{g(W, \cos\theta)}{\alpha_0 - \cos\theta} + f(W, \cos\theta) \tag{4}$$

where g, f are analytic in the same region of the $\cos\theta$ plane as M except that they have no singularity at $\cos\theta = \alpha_0$. The first term on the right-hand side of eq. (4) is just the renormalised Born approximation amplitude, so that $g(W, \cos\theta)$ can be determined as a function of W, θ, and the renormalised coupling constants. The explicit form of g will be considered in the next section.

If $f(W, z) \to 0$ as $|z| \to \infty$ we may write

$$f(W,z)=\int_{-\infty}^{\beta_1}\frac{\rho_-(W,z')}{z'-z}\,\mathrm{d}z'+\int_{\alpha_1}^{\infty}\frac{\rho_+(W,z')}{z'-z}\,\mathrm{d}z'+\frac{\rho_0(W)}{\beta_0-z}\,. \qquad (5)$$

In general, the usual assumptions from which the dispersion relations are derived would only allow us to conclude that eq. (5) is correct to within an additive polynomial in z, with coefficients which are functions of W, provided a suitable convergence factor $|z'|^{-n}$ is included in the integrands.

In order to discuss the relative importance of the various terms entering eqs. (4) and (5) it is useful to consider the values of α_0, α_1, β_0, β_1 for various values of W, or of the energy ω_L of the pion in the laboratory system of the nucleon. These values are given in table 1. We see from this table that the distance of α_0 from the forward direction $\cos\theta = 1$ is less than that of β_0

TABLE 1

Values of the parameters α_0, α_1, β_0, β_1, as defined in section 2, for various values of the pion energy ω_L in the nucleon laboratory system; Y is the produced hyperon, and Y′ the intermediate hyperon, if any.

ω_L	α_0		α_1		$-\beta_0$			$-\beta_1$	
Y	Λ	Σ	Λ	Σ	Λ	Σ	Σ	Λ	Σ
Y′					Σ	Λ	Σ	Σ	Σ
1.1	1.92	2.57	2.17	3.25	4.5	5.3	6.2	5.4	7.6
1.9	1.24	1.27	1.40	1.43	2.15	2.0	2.2	2.45	3.2
4.5	1.07	1.07	1.11	1.12	1.38	1.37	1.52	1.48	1.7

by at least a factor 5, while α_0 and β_0 are about equidistant from $\cos\theta = 0$. If the continuum contribution on the right-hand side of eq. (5) from values of $z' > \alpha_1$ is mainly from points not too close to α_1, and if f, g have the same order of magnitude over the physical range, then we expect that the pole in the production amplitude at $\cos\theta = \alpha_0$ will give a contribution which will make the K-meson production strongly peaked in the forward direction †.

It will be a poor approximation in the physical region to expand this contribution in a power-series in $\cos\theta$, especially near $\cos\theta = 1$. However, as we see from table 1, it will be more reasonable to expand the remaining part of the production amplitude in powers of $\cos\theta$. We may do this, since from our basic analyticity conjecture we may represent $f(W, \cos\theta)$, to any required degree of approximation, in the region $|\cos\theta| < \alpha$, by a suitable polynomial in $\cos\theta$. Here α determines the region of analyticity of $f(W, \cos\theta)$ and our analyticity conjecture requires that $\alpha > \alpha_0$. We might remark here that even if we do not assume the analyticity property we may still approximate any suitably well-behaved function $f(W, \cos\theta)$ by a polynomial in

† This result has already been noted [3, 4]).

$\cos\theta$ in the region $|\cos\theta| < \alpha$, or even only in $|\cos\theta|<1$. Restricting this approximation to $|\cos\theta|<1$ would allow us to determine parity assignments, but not to obtain coupling constants in the manner used for the pion-nucleon case [2]).

The representation for the production amplitude which we have arrived at was used in ref.[2]) by continuing the function $F(W, \cos\theta) = (\alpha_0 - \cos\theta)^2 (\mathrm{d}\sigma/\mathrm{d}\Omega)$ from values of $\cos\theta$ near 1 to $\cos\theta = \alpha_0$. This gives $g^2(W, \alpha_0)$, and hence a value for the pion-nucleon coupling constant. We may go through the same procedure in our case. It is difficult to obtain the parity assignments for the strange particles in this manner, since both these parity assignments and the unknown magnitude of certain coupling constants occur in $g^2(W, \alpha_0)$. We shall obtain our parity assignments by fitting the angular distributions arrived at above to the experimental distributions. It will be possible afterwards to obtain the coupling constants as in the pion-nucleon case, though not enough data is available to do this yet.

What has been developed above for associated production by pions will apply in a similar manner to production by photons and to scattering processes involving strange particles. In particular, regions of analyticity in the $\cos\theta$-plane can be derived as above, and residues and angular distributions discussed.

It has not yet been possible to prove the conjecture on which we are basing our method, using similar properties and tools to those used in proving dispersion relations. This is very likely because our present methods are inadequate. Our conjecture seems reasonable from the heuristic arguments given at the beginning of this section, so the lack of proof need not prevent us from having reasonable faith in our conjecture.

The conjectured cut plane of analyticity is more than we need for our present discussion. We only need require that eq. (4) holds for $f(W, \cos\theta)$ analytic in some region including $|\cos\theta| \leqq \alpha_0$, and is well approximated for $\cos\theta \approx 1$ by a polynomial in $\cos\theta$. We require also that the first term on the right-hand side of (4) be the renormalised Born approximation. Although this modified analyticity conjecture has not yet been proved it seems to have much more chance of being so. We will assume it here, without further comment.

3. Parity Assignments

The processes which we will consider now in detail are

$$\pi^- + \mathrm{P} \to \mathrm{K}^0 + \Lambda^0 \tag{6}$$

$$\to \mathrm{K}^0 + \Sigma^0 \tag{7}$$

$$\to \mathrm{K}^+ + \Sigma^- \tag{8}$$

$$\pi^+ + \mathrm{P} \to \mathrm{K}^+ + \Sigma^+. \tag{9}$$

For process (6), $Y = \Lambda$, $Y' = \Sigma$. From the analyticity conjecture of the previous section we may write the production amplitude for the process (6) as

$$M(W, \cos\theta) = (2\pi)^{-2}\delta^4(p+k-p'-q) \cdot \left(\frac{M\Lambda}{4p_0p'_0q_0k_0}\right)^{\frac{1}{2}} \frac{1}{2kq}\left[\frac{fg_\Lambda \bar{u}_\Lambda(p')\Gamma u_N(p)}{\alpha_0-\cos\theta} + T(W, \cos\theta)\right]. \quad (10)$$

We are assuming that the spin of the Λ and Σ is $\frac{1}{2}$ and that of the K is zero. Γ is 1 or γ_5 according as $P_\Lambda(K_+)$ (the K^+ parity with respect to the ΛN system) is even or odd. We expect $T(W, \cos\theta)$ to be a polynomial in $\cos\theta$ which depends on the spin direction of the Λ and N. f is the renormalised coupling constant arising in the $(KK\pi)$ interaction, and is defined by

$$f = (4q_0k_0)^{\frac{1}{2}}\langle K^0, k|j_\pi(0)|K^+, q\rangle \quad (11)$$

where the right-hand side of eq. (11) is evaluated for $(k-q)^2 = -\mu^2$. In a Lagrangian language the interaction $f_0\bar{\phi}_{K^0}\phi_{K^+}\phi_\pi$ will have f as its renormalised coupling constant. We will not confine ourselves here to this language. g is the $(K\Lambda P)$ coupling constant, defined by

$$g_\Lambda = \left(\frac{p_0p'_0}{M\Lambda}\right)^{\frac{1}{2}}\langle \Lambda, p'|j_{K^+}(0)|P, p\rangle \quad (12)$$

for $(p'-p)^2 = -\kappa^2$. We require invariance under space rotations of the matrix elements corresponding to the $(KK\pi)$ interactions, so that in order that f is not zero we must have odd relative parity $p(K)$ of K^+ relative to K^0. That this is so has been conjectured before [4]) †. What we propose to show is that this result follows unambiguously from present experimental data.

In order to do this, we use the fact that from eq. (10) the differential cross section for process (6), at fixed energy W, and multiplied by $(\alpha_0-\cos\theta)^2$, is given by

$$\begin{aligned} F_\Lambda(W, \cos\theta) &= (\alpha_0-\cos\theta)^2\frac{d\sigma}{d\Omega} \\ &= K(W)\{f^2g_\Lambda^2 \operatorname{Tr}[(i\gamma p'-\Lambda)\Gamma(i\gamma p-M)\bar{\Gamma}]/8\Lambda M \\ &+ fg_\Lambda(\alpha_0-\cos\theta)\tfrac{1}{2}\Sigma \operatorname{Re}[\bar{u}_\Lambda(p')\Gamma u_N(p)\overline{T(W, \cos\theta)}] \\ &+ \tfrac{1}{2}\Sigma(\alpha_0-\cos\theta)^2|T(W, \cos\theta)|^2\}. \end{aligned} \quad (13)$$

In eq. (13) $K(W)$ is a known function depending on the energy only, and Σ denotes summation over the spin direction of the Λ and nucleon. We denote $\frac{1}{2}\Sigma|T(W, \cos\theta)|^2$ by $P(W, \cos\theta)$ and $\frac{1}{2}\Sigma \operatorname{Re}[\bar{u}_\Lambda(p')\Gamma u_N(p)\overline{T(W, \cos\theta)}]$ by $Q(W, \cos\theta)$. Then P and Q will be polynomials in $\cos\theta$.

The trace term is $4(-pp'\pm\Lambda N)$, where the plus sign is for $\Gamma = 1$ and the minus sign is for $\Gamma = \gamma_5$. In the centre of mass system this term has an angular dependence $4(p'_0p_0-kq\cos\theta\pm\Lambda N)$. To obtain an idea of the size of

† The author would like to thank J. Hamilton for bringing this to his attention.

this contribution we note that at $\omega_L = 1.9$ GeV this term becomes $4\Lambda N$ $(1.58-0.53\cos\theta\pm1)$. Thus for $\Gamma = \gamma_5$ this term becomes equal to $2.12\Lambda N$ $(1.09-\cos\theta)$, and at $\cos\theta = \alpha_0$ takes the value $-0.32\Lambda N$. For $\Gamma = 1$ the trace term is $2.12\Lambda N(4.9-\cos\theta)$ and at $\cos\theta = \alpha_0$ takes the value $7.9\Lambda N$, so is greater than the term for $\Gamma = \gamma_5$ by a factor 24. Thus this trace term will give a quite different angular dependence to $F(W, \cos\theta)$ for the two possibilities of Γ, unless $fg_\Lambda \ll t(W)$, where $t(W)$ is an average value of $|T(W, \cos\theta)|$ for $\cos\theta$ near 1. Only if this inequality is satisfied will $F_\Lambda(W, \cos)$ be very nearly proportional to $(\alpha_0-\cos\theta)$, and the angular dependence for $\Gamma = 1$ or γ_5 have any chance of being similar. In the alternative case we expect $F(W, \cos\theta)$ not to go as close to the point $\cos\theta = \alpha_0$ for $\Gamma = 1$ as for $\Gamma = \gamma_5$. We cannot make this argument very precise since to know what we mean by "near the point $\cos\theta = \alpha_0$" it is necessary to know the values of f and g_Λ. To get around this difficulty we shall use the actual values of $F_\Lambda(W, \cos\theta)$ for a range of values near $\cos\theta = 1$, as we mentioned earlier.

Using experimental values [5]) of $d\sigma/d\Omega$ we have plotted in fig. 1 the curve

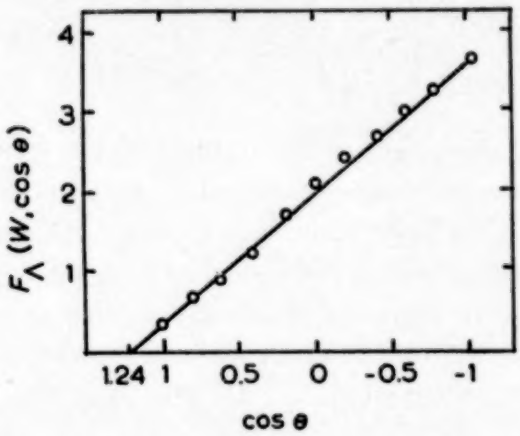

Fig. 1. Values of $F_\Lambda(W, \cos\theta) = (\alpha_0-\cos\theta)^2\,(d\sigma/d\Omega)$, using values of the Λ^0 production angular distribution $d\sigma/d\Omega$ interpolated from those of ref.[5]). The units are arbitrary, and no experimental errors can be given. The line $F_\Lambda = 1.73(\alpha_0-\cos\theta)$ is drawn for comparison.

of $F_\Lambda(W, \cos\theta)$ against $\cos\theta$. We see that the curve is very close to a straight line through $\cos\theta = \alpha_0$. It is necessary to point out here that owing to the scarcity of data not too much reliance may be placed on the exactness of this curve. However, it may be expected from the general shape of the forward peaking of the K^0 that the data should lie reasonably close to the curve of fig. 1. Further data are necessary to make our conclusions more certain, and it is hoped that this data, particularly in the forward direction, will be forthcoming. The conclusions we derive here are all subject to possible revision from further data.

The first conclusion we can draw from the shape of $F_\Lambda(W, \cos\theta)$ in fig. 1 is that neither f nor g_Λ are zero. For if either $f = 0$ or $g_\Lambda = 0$ then

from eq. (13) we have

$$F_A(W, \cos\theta) = (\alpha_0 - \cos\theta)^2 P(W, \cos\theta). \tag{14}$$

The curve of fig. 1 would seem to rule out this possibility. In particular we do not expect higher than P-wave K^0 production at $\omega_L = 1.9$ GeV; therefore, $P(W, \cos\theta) = a + b\cos\theta + c\cos^2\theta$. It is not possible to fit $F(W, \cos\theta)$ at all closely by a curve of the form $(\alpha_0 - \cos\theta^2)(a + b\cos\theta + c\cos^2\theta)$ for any a, b, c. Addition of higher angular momentum states does not alter this result, since what is required is to fit the curve of fig. 1 by a curve with a second order zero at $\cos\theta = \alpha_0$, which is evidently very difficult unless very high powers of $\cos\theta$ are included in $P(W, \cos\theta)$.

We can never show that the experimental values for $F_A(W, \cos\theta)$ will be *exactly* on a straight line whose continuation will pass through α_0. Only if we could, would we have shown that the right hand side of eq. (14), with $P(W, \cos\theta)$ analytic at $\cos\theta = \alpha_0$, cannot fit our experimental curve for $F_A(W, \cos\theta)$. The spread of experimental values will always allow us to continue $F_A(W, \cos\theta)$ from the physical region to $\cos\theta = \alpha_0$ so as to be fitted by the right hand side of eq. (14). This problem of continuation from the physical region, when our values there are only given within an experimental error, occurs also in the application of dispersion relations. We can only treat it here by conjecturing that the curve of fig. 1 *really is* a straight line whose continuation goes through $\cos\theta = \alpha_0$. The curve of fig. 1 is strong evidence in support of our conjecture, so we accept it here as correct. Future data will give a more stringent test of our conjecture. Hence we conclude that there is strong evidence, from the present data, that $p(K)$ is odd.

We will now try to determine $P_A(K_+)$. Let us assume first that $P_A(K_+)$ is even. Then

$$F_A(W, \cos\theta) = K(W)\{0.3f^2 g_A^2(4.9 - \cos\theta) + fg_A(\alpha_0 - \cos\theta)Q(W, \cos\theta) + (\alpha_0 - \cos\theta)^2 P(W, \cos\theta)\}. \tag{15}$$

We denote the polynomial expansions of $P(W, \cos\theta)$ and $Q(W, \cos\theta)$ by $\sum a_n \cos^n\theta$ and $\sum b_n \cos^n\theta$. In order that $F_A(W, \cos\theta)$ be very closely represented by a term proportional to $\alpha_0 - \cos\theta$ over the physical region we require

$$fg_A \ll |b_0| \tag{16}$$

$$|b_n| \ll |b_0| \quad \text{for} \quad n = 1, 2, \ldots \tag{17}$$

$$|a_m| \ll fg_A|b_0| \quad \text{for} \quad m = 0, 1, 2, \ldots. \tag{18}$$

Hence

$$|a_0| \ll |b_0|^2. \tag{19}$$

Using invariance of the strong interactions under the homogeneous Lorentz

group and elementary algebra it is possible to show that

$$|a_0| > 0.75\,|b_0|^2. \tag{20}$$

Hence we get a contradiction between eqs. (19) and (20). The precise meaning of the inequality $\ll$ entering into eqs. (16) to (19) cannot be made at all precise with present data. Our argument can only be used to indicate that it is difficult to fit the data with $P_\Lambda(\mathrm{K}_+)$ even. In principle our argument can be used with more comprehensive data to show the possibly complete inconsistency of even $P_\Lambda(\mathrm{K}_+)$ with the data.

We might notice that our rough order of magnitudes argument of eqs. (16) to (20) gives us the inequality $fg_\Lambda \gg t(W)$. In particular if we assume that $t(W)$ is of the same order of magnitude as Gg_Σ (arising from the process of fig. 2), where g_Σ and G are the (KΣP) and ($\pi\Sigma\Lambda$) coupling constants, then we have $fg_\Lambda \gg Gg_\Sigma$. We may even expect $f \gg G$, since we may deduce that $g_\Lambda \approx g_\Sigma$ from the similarity of the shape of $F(W, \cos\theta)$ for Σ^0 and Λ^0 production. As we have already remarked, it will be necessary to obtain more data before a more exact meaning can be given to the sign $\ll$. We

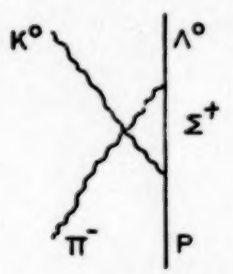

Fig. 2. Diagram of the process which is expected to give rise to a term $O(Gg_\Sigma)$ in the production amplitude for process (6).

cannot hope to obtain anything precise in this way, however, since we have no method to determine the way in which the various coupling constants occur in $T(W, \cos\theta)$, assuming, say, the usual interactions. We may still expect from our argument that f is at least the same order as the usual meson-baryon coupling constants [6]).

If we assume that $P_\Lambda(\mathrm{K}_+)$ is odd then the inequality of eq. (16) is no longer required, and the contradiction disappears. Hence we conclude that the angular distribution can be fitted more easily by $P_\Lambda(\mathrm{K}_+)$ odd than by $P_\Lambda(\mathrm{K}_+)$ even [7]).

Our method can also be applied to the Σ-production processes (7), (8), and (9). The data reported at the 1957 Rochester Conference for associated production of Λ^0 and Σ^0 at 1.1 GeV pion energy [8]) leads one to the conclusion that K^+ has the same parity with respect to the (Σ^0N) system as with respect to the (Λ^0N) system, so $P_\Lambda(\mathrm{K}_+) = P_\Sigma(\mathrm{K}_+)$. This is because the forward peaking of the K^0 has the same general appearance in both Λ^0 and Σ^0 production. A quantitative argument that $P_\Sigma(\mathrm{K}_+)$ is odd using the data

as we did for $P_{\Lambda}(K_+)$ cannot be given from the data of ref.[8]), the main reason being that now $\alpha_0 = 2.57$. Thus the contribution to the production cross-section from the Born term for Σ^0 production compared to the remaining contributions will be expected to be much less at 1.1 GeV that at 1.9 GeV, for Λ^0 production, by a factor at least of order 6 if $P_{\Lambda}(K_+) = P_{\Sigma}(K_+)$ and by one at least of order 11 if otherwise. Hence we may expect the contribution of the last term on the right hand side of eq. (15) to be important now. Since S-wave production is expected to predominate, we may expect $F_{\Sigma}(W, \cos\theta) = (\alpha_0 - \cos\theta)^2(d\sigma/d\Omega) \approx (\alpha_0 - \cos\theta)(a + b\cos\theta)$, where a, b are some constants and $d\sigma/d\Omega$ is the K^0 angular distribution. Only if the $|F|^2$ term is small in comparison to the interference term do we hope to determine $P_{\Sigma}(K_+)$ and $p(K)$ unambiguously. This corresponds to $|b/a| \ll 1$. The present data can be roughly fitted by $a/b \approx \frac{1}{5}$, though more complete data are necessary. Even if we had such data it may still be difficult to pick out the Born term contribution at this low energy. It is hoped that data for Σ production at $\omega_L \gtrsim 2.0$ GeV will be obtained so that we may reach a more reliable conclusion about $P_{\Sigma}(K_+)$.

In process (8) the Born term contribution giving a pole in the amplitude at $\cos\theta = \alpha_0$ vanishes by charge conservation, so we only have singularities at $\cos\theta = \beta_0$. The discussion of the previous section about the contribution from the pole at $\cos\theta = \alpha_0$ can be applied to our present case. Hence we expect that the Σ^- production is peaked in the forward direction; this peaking arising from the Born terms of fig. 3 [3]). We denote the values of β_0

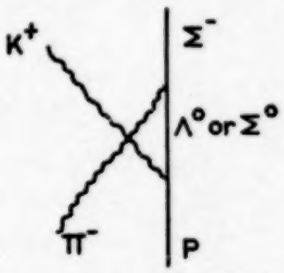

Fig. 3. Diagram of the process giving the Born term expected to give forward peaking of Σ^- in process (8).

corresponding to the intermediate hyperon Y′ being Λ^0 or Σ^0 by $\beta_0(\Lambda)$, $\beta_0(\Sigma)$. A plot of $F_{\Sigma}(W, \cos\theta) = (\beta_0(\Lambda) + \cos\theta)^2(d\sigma_-/d\Omega)$ against $\cos\theta$ is shown in fig. 4, where $d\sigma_-/d\Omega$ is the Σ^- production differential cross section as taken from ref. [8]). Although our extension of the data to forward angles may be unreliable we see that $F_{\Sigma}(W, \cos\theta)$ appears to be quite flat near the forward direction. For either value of $P_{\Sigma}(K_+)$ we expect the Born term to give a contribution to the angular distribution proportional to $(1 + 0.02\cos^2\theta)(\beta_0(\Lambda) + \cos\theta)^{-2}$, so we may explain the flatness of $F_{\Sigma}(W, \cos\theta)$ by assuming the Born term to give a majority of the contribution, at least in the

group and elementary algebra it is possible to show that

$$|a_0| > 0.75\,|b_0|^2. \tag{20}$$

Hence we get a contradiction between eqs. (19) and (20). The precise meaning of the inequality $\ll$ entering into eqs. (16) to (19) cannot be made at all precise with present data. Our argument can only be used to indicate that it is difficult to fit the data with $P_\Lambda(\mathrm{K}_+)$ even. In principle our argument can be used with more comprehensive data to show the possibly complete inconsistency of even $P_\Lambda(\mathrm{K}_+)$ with the data.

We might notice that our rough order of magnitudes argument of eqs. (16) to (20) gives us the inequality $fg_\Lambda \gg t(W)$. In particular if we assume that $t(W)$ is of the same order of magnitude as Gg_Σ (arising from the process of fig. 2), where g_Σ and G are the (KΣP) and ($\pi\Sigma\Lambda$) coupling constants, then we have $fg_\Lambda \gg Gg_\Sigma$. We may even expect $f \gg G$, since we may deduce that $g_\Lambda \approx g_\Sigma$ from the similarity of the shape of $F(W, \cos\theta)$ for Σ^0 and Λ^0 production. As we have already remarked, it will be necessary to obtain more data before a more exact meaning can be given to the sign $\ll$. We

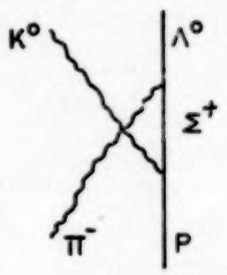

Fig. 2. Diagram of the process which is expected to give rise to a term $O(Gg_\Sigma)$ in the production amplitude for process (6).

cannot hope to obtain anything precise in this way, however, since we have no method to determine the way in which the various coupling constants occur in $T(W, \cos\theta)$, assuming, say, the usual interactions. We may still expect from our argument that f is at least the same order as the usual meson-baryon coupling constants [6]).

If we assume that $P_\Lambda(\mathrm{K}_+)$ is odd then the inequality of eq. (16) is no longer required, and the contradiction disappears. Hence we conclude that the angular distribution can be fitted more easily by $P_\Lambda(\mathrm{K}_+)$ odd than by $P_\Lambda(\mathrm{K}_+)$ even [7]).

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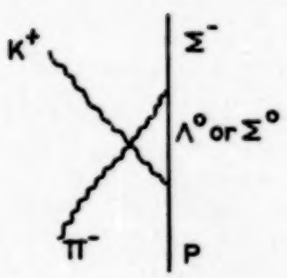

Fig. 3. Diagram of the process giving the Born term expected to give forward peaking of Σ^- in process (8).

corresponding to the intermediate hyperon Y′ being Λ^0 or Σ^0 by $\beta_0(\Lambda)$, $\beta_0(\Sigma)$. A plot of $F_\Sigma(W, \cos\theta) = (\beta_0(\Lambda) + \cos\theta)^2(d\sigma_-/d\Omega)$ against $\cos\theta$ is shown in fig. 4, where $d\sigma_-/d\Omega$ is the Σ^- production differential cross section as taken from ref. [8]). Although our extension of the data to forward angles may be unreliable we see that $F_\Sigma(W, \cos\theta)$ appears to be quite flat near the forward direction. For either value of $P_\Sigma(K_+)$ we expect the Born term to give a contribution to the angular distribution proportional to $(1 + 0.02\cos^2\theta)(\beta_0(\Lambda) + \cos\theta)^{-2}$, so we may explain the flatness of $F_\Sigma(W, \cos\theta)$ by assuming the Born term to give a majority of the contribution, at least in the

forward direction. This is to be compared with the Σ^0 production, where the contribution in the backward direction from the Born term is not all-important. The explanation of this result may be in the fact that for either value of $P_\Sigma(K_+)$ the Born term in Σ^- production behaves as $(\beta_0+\cos\theta)^{-2}$, while for odd $P_\Sigma(K_+)$ the Born term in Σ^0 production is essentially reduced to $(\alpha_0+\cos\theta)^{-1}$, and so does not have such a high singularity. This would seem to be a further argument in favour of odd $P_\Sigma(K_+)$.

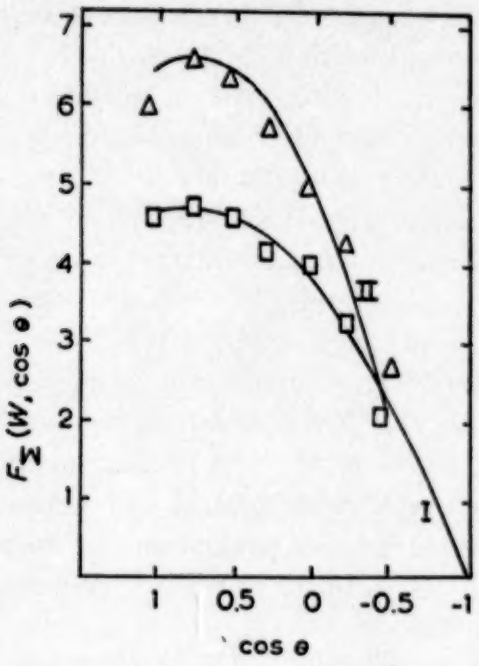

Fig. 4. Values of $F_\Sigma(W, \cos\theta) = (\beta_0(Y')-\cos\theta)^2\,(d\sigma/d\Omega)$ using values of the Σ^- production angular distribution $d\sigma/d\Omega$ interpolated from those of ref.[8]). The points □ are for $Y' = \Lambda$, and △ for $Y' = \Sigma$. The units are arbitrary, and no experimental errors can be given. Curve I is $F_\Sigma = 1.54(2.5-\cos\theta)\,(1+\cos\theta)$ and Curve II is $F_\Sigma = 2.94(2.25-\cos\theta)(0.75+\cos\theta)$, drawn for comparison.

Some information about the $(\pi\Sigma\Lambda)$, $(\pi\Sigma)$, $(KP\Lambda)$, and $(KP\Sigma)$ coupling constants may also be derived from more complete data for Σ^- production. This is because the Σ^- production amplitude contains the two terms with singularities at $\beta_0(\Lambda)$ and $\beta_0(\Sigma)$ and residues which are known except for the products Gg_Λ and $G_\Sigma g_\Sigma$ respectively, where G_Σ is the $(\pi\Sigma)$ coupling constant. If we form $(\beta_0(\Lambda)+\cos\theta)^2(d\sigma_-/d\Omega)$ and $(\beta_0(\Sigma)+\cos\theta)^2(d\sigma_-/d\Omega)$, and if one or other of these curves is a constant near the forward direction, evidently the corresponding product of coupling constants is likely to be much larger than the other. In fig. 4 both these curves are drawn at $\omega_L = 1.1$ GeV from the data of ref.[8]). These curves would seem to show, with the above argument, that $Gg_\Lambda > G_\Sigma g_\Sigma$. We cannot put any faith in this inequality because of lack of data.

We finally come to the process of eq. (10). We expect there to be a forward peaking in the K-production from the Born term in fig. 5. Experimentally there is forward peaking of the Σ^+ production with roughly the same angular shape as in the Σ^- production [8]). We can only obtain such a result provided the matrix element $\langle \Sigma^+ | j_{K^0} | P \rangle$ is very small compared to other matrix elements entering into process (10). This may be obtained in a number of

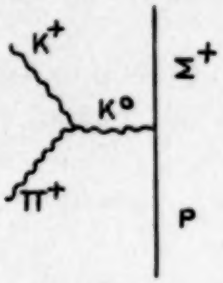

Fig. 5. Diagram of the process which is expected to cause forward peaking of the produced K^+ in process (9).

ways. One is to require that K^+ and K^0 form a doublet, at least so that $\langle \Sigma | j_{K^0} | P \rangle$ and $\langle \Sigma | j_{K^+} | P \rangle$ transform in a similar manner under the extended Lorentz group. Since we have found that $p(K)$ is almost certainly odd, at

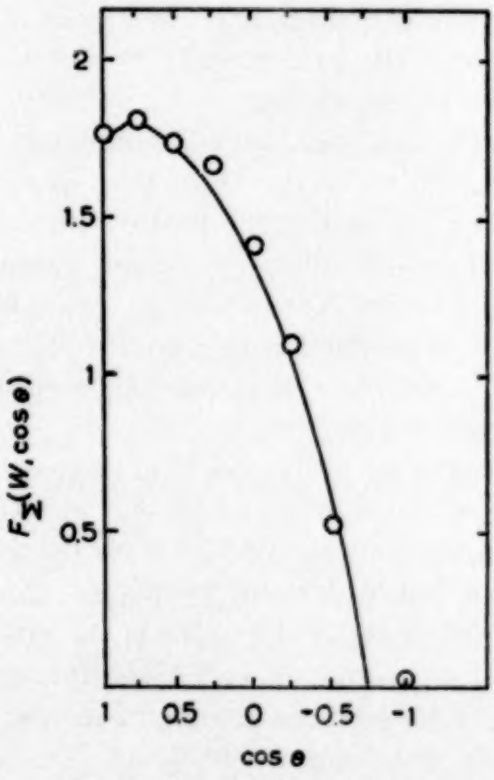

Fig. 6. Values of $F_\Sigma(W, \cos\theta) = (\beta_0(\Lambda) - \cos\theta)^2 \, (d\sigma/d\Omega)$ using values of the Σ^+ production angular distribution $d\sigma/d\Omega$ interpolated from those of ref.[8]). The units are arbitrary and no experimental errors can be given. The curve $F_\Sigma = 0.79(2.25 - \cos\theta)\ (0.75 + \cos\theta)$ is drawn for comparison.

least one of these matrix elements must vanish. Our discussion of Σ^0 production led us to conclude that $\langle\Sigma|j_{K^+}|P\rangle$ is not zero, so that $\langle\Sigma^+|j_{K^0}|P\rangle = 0$. In fig. 6 we show the curve of $(\beta_0(\Lambda)+\cos\theta)^2(d\sigma/d\Omega)$, using the data of ref. [8]). The flattering in the forward direction would seem to suggest the presence of the double pole at $\cos\theta = -\beta_0(\Lambda)$. We certainly need further data for $\omega_L \gtrsim 2.0$ GeV to determine if the backward peaking in K^+ production is really due to this pole at $\cos\theta = \beta_0$ before the vanishing of $\langle\Sigma^+|j_{K^0}|P\rangle$ is forced on us.

To close this section it may be useful to remark that, if better data for Λ and Σ production agree with our present parity assignments for $p(K)$, $P_\Lambda(K_+)$, $P_\Sigma(K_+)$, by continuing $F_\Lambda(W, \cos\theta)$ and $F_{\Sigma^0}(W, \cos\theta)$ to $\cos\theta = \alpha_0$ we can obtain numerical values for fg_Λ and fg_Σ.

4. Other Processes

There are two other types of processes, namely associated production by photons and elastic scattering, which may give further information concerning parity assignments and sizes of coupling constants by similar methods to those developed in the previous sections.

For associated production by photons we expect a pole of second order in the Λ^0 production angular distribution at $\cos\theta = \alpha_0 = 1/v_K$ where v_K is the velocity of the K-meson. The present data [10]) at 1 GeV give a quadratic dependence of $(\alpha_0-\cos\theta)^2(d\sigma/d\Omega)$ on $\cos\theta$, probably for the same reason as for the quadratic behaviour in $\cos\theta$ for Σ^0 production by pions at 1.1 GeV. That is, α_0 is too large to make the Born term give the majority of the production cross-section, at least in the forward direction. Associated production by photons at higher energies, at least above 1.5 GeV, does not seem possible at present, so we cannot hope to obtain much useful information from this process. If production by photons of energy at least 2 GeV were to become possible, we might go through an analysis of the data similar to that in the previous section.

Charge exchange scattering of K^- on P may give further information, provided the incoming K momentum is high enough. At presently available values of laboratory momentum ≈ 400 MeV/c the Born pole is at $\cos\theta \approx 2.6$, so we do not expect too much peaking from this term, especially since we have a damping term behaving as $(1-\cos\theta)$ in the numerator of this term. It may be that for momentum $\gtrsim 1$ GeV/c the interference term, with a simple pole at $\cos\theta \lesssim 1.35$, becomes predominant, and so we may analyse the data as we did for Λ^0 production by pions. This exchange scattering process will give us further information on the $(KK\pi)$ interaction, and in particular continuation of $(\alpha_0 - \cos\theta)^2(d\sigma/d\Omega)$ to $\cos\theta = \alpha_0$ will give us an absolute value for f. Since we can, in principle, obtain fg_Λ and fg_Σ by the

methods developed in the previous section, we may obtain values for g_Λ and g_Σ. However, no information on $P_\Lambda(K_+)$ or $P_\Sigma(K_+)$ can be obtained by our methods from this process.

5. Conclusion

Our parity assignments and other results are:

(1) $p(K)$ and $P_\Lambda(K_+)$ are both odd from Λ^0 production by pions, so that K^+ and K^0 have opposite parity, and K^+ has odd parity with respect to the (ΛN) system;

(2) $fg_\Lambda \gg Gg_\Sigma$, where f, G, g_Λ, g_Σ are the $(KK\pi)$, $(\pi\Sigma\Lambda)$, $(K\Lambda N)$, $(K\Sigma N)$ coupling constants;

(3) $P_\Sigma(K_+)$ is odd from Σ^0 production by pions, so that relative parity of Σ and Λ is even;

(4) from Σ^+ production by positive pions the pair (K^+, K^0) forms a doublet with respect to the (ΛN) system, at least for transformation behaviour under the extended Lorentz group.

All of these conclusions are based upon too few data to be inescapable, although we may have considerable confidence that $p(K)$ is odd from the discussion of Λ^0 production by pions. The remaining conclusions may require alteration when more comprehensive data are forthcoming.

Thus our main conclusions are twofold, namely that $p(K)$ is odd, and that more data are required. More data are especially required on the angular distribution of Λ^0 and Σ in associated production by pions, particularly near the angle of maximum production, for pion energy at least 2 GeV. With such data our discussion shows that certain parity assignments and coupling constants may then be determined, independently of any approximation schemes such as perturbation theory, or of any particular Lagrangian scheme. Although the analyticity method we have developed above is limited in its description of the details of the strong interactions, as are the dispersion relations, yet our method gives, and will give in the future, definite answers to questions about the parity of elementary particles and the magnitudes of their coupling constants.

The author would like to thank S. B. Treiman for a helpful discussion and the University of Maryland for its hospitality during part of the time when this work was being completed.

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ISOTOPE SHIFT AND CHANGES OF NUCLEAR RADIUS

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Abstract: A new derivation of the expression for the nuclear charge distribution dependent isotope shift (I.S.) is given. This derivation shows clearly what approximations are made. The result, essentially the same as already obtained previously by the author, is as simple as the perturbation theory expression for the I.S. but is independent of the limitations of this. It depends on the charge distribution as a whole only through the root-mean-square radius, $\langle r^2\rangle^{\frac{1}{2}}$ and on the difference of charge distribution between isotopes only through $\delta\langle r^2\rangle^{\frac{1}{2}}$. The implications of the electron scattering results for the I.S. are considered both for its dependence on $\langle r^2\rangle^{\frac{1}{2}}$ *and* on $\delta\langle r^2\rangle^{\frac{1}{2}}$. A renewed discussion is given of the effect of nuclear compressibility on the I.S. first considered by Wilets, Hill and Ford. For this the dependence of the symmetry energy on the density is shown to be as important as the Coulomb energy. Thus the I.S. cannot be uniquely related to the compressibility coefficient of nuclear matter but may now however be used to obtain information about the density dependence of the symmetry energy. The implications of the I.S. data for changes of radius between isotones and between nuclei along the mass valley are also briefly considered.

1. Introduction

The spectroscopic isotope shift arises predominantly from a difference of mass and a difference of charge distribution between isotopes. The latter effect is predominant for heavier nuclei ($A \gtrsim 100$) and is the only one of interest for nuclear structure. In this paper it is always to be understood that any reference to the isotope shift (I.S.) is only to the charge distribution effect.

Apart from purely electronic factors the I.S. depends on the nuclear charge distribution and its difference between isotopes. Since the nuclear charge distribution is now known rather accurately from the electron scattering experiments, this knowledge can be used to extract information from the I.S. data about the differences of charge distribution between isotopes. In fact the I.S. is almost the only *direct* source of information available at present about the changes of charge distribution between neighbouring nuclei and this in fact only for differences between isotopes, i.e. due to a change of the number of neutrons only. For recent reviews of the I.S. see Kopfermann [1]) and Breit [2]).

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The substance of the present paper falls into three parts. In section 2 a new derivation of the theoretical expression for the I.S. will be given in order to show quite clearly how the charge distribution and its difference between isotopes enter into this expression and the nature of the approximations made in obtaining this. The result is essentially the same as that already obtained previously by the author in ref.[3]) (to be referred to as I), which however does not seem to be well known. A particularly important feature of this result is its general character.

In section 3 the implications of the electron scattering results will be considered using the results of section 2, not only, as is usually done, for the dependence of the I.S. on the nuclear charge distribution as a whole *but also* for the dependence on the differences of charge distribution.

In section 4 a renewed discussion is given of the effect of nuclear compressibility on the I.S. first considered by Wilets, Hill and Ford [4]). In particular not only the Coulomb energy but also the dependence of the symmetry energy on the nuclear density, which was neglected by these authors, will be shown to play a vital role. The resulting implications of the I.S. data, especially with reference to this dependence, will be discussed taking into account the results obtained in section 3, and also with reference to the large values of the compressibility coefficient indicated by recent investigations [5, 6]). The implications of the I.S. data on changes of radius between isotones and also between nuclei along the bottom of the mass valley will also be briefly considered.

2. Theory of the Isotope Shift

In previous derivations of the I.S. the difference of energy ΔE between an electron in the electrostatic field of a point charge and in that of an extended charge distribution with the same total charge is first obtained and only then is the I.S. $\delta(\Delta E)$ obtained as the change of ΔE due to the difference of charge distribution between the isotopes. In the following derivation, which is based on a method due to Broch [7]), the first step is avoided and the I.S. is obtained directly as the difference of electron energy between isotopes. This procedure has the advantage not only of simplicity and directness but also of showing quite clearly what approximations are made in obtaining the final expression. To avoid complications of notation the I.S. will be denoted by δE instead of, as is customary, by $\delta(\Delta E)$. It is convenient to use the quantities

$$x = 2Zr/a_0, \qquad a = Z\alpha$$
$$\varepsilon = E/mc^2, \quad U(r) = V(r)/mc^2$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius and $\alpha = e^2/\hbar c$; $V(r)$ is the potential energy (assumed spherically symmetric) of the electron in the field of the

nuclear charge distribution. The radial Dirac equations may then be written in the form

$$\frac{\mathrm{d}f_k}{\mathrm{d}x} = \frac{k}{x} f_k + \frac{1}{2a}(1+U-\varepsilon) g_k$$
$$\frac{\mathrm{d}g_k}{\mathrm{d}x} = \frac{1}{2a}(1-U+\varepsilon) f_k - \frac{k}{x} g_k \qquad (1)$$

where

$$k = -(l+1) \quad \text{for} \quad j = l+\tfrac{1}{2}$$
$$= \quad l \quad\quad \text{for} \quad j = l-\tfrac{1}{2}$$

and f_k, g_k correspond to the small and large components of the Dirac wave-function respectively.

Quantities referring to the heavier isotope will be distinguished by a dash. Thus $\delta U = U'-U$ is the difference of the electron potential energy between the heavier and lighter isotope and $\delta\varepsilon = \varepsilon'-\varepsilon$ is the corresponding I.S. An immediate consequence of (1) is then

$$2a \frac{\mathrm{d}}{\mathrm{d}x}(f'_k g_k - f_k g'_k) = (\delta U - \delta\varepsilon)(f_k f'_k + g_k g'_k). \qquad (2)$$

We consider a point x_0 corresponding to a distance of the order of magnitude of the nuclear radius and such that x_0 is outside the charge distributions of both isotopes, i.e. such that $\delta U = 0$ for $x \geqq x_0$. Then integrating (2) from $x = x_0$ to $x = \infty$ and putting $\delta f_k = f'_k - f_k$, $\delta g_k = g'_k - g_k$, one gets

$$\delta\varepsilon = 2a(g_k \delta f_k - f_k \delta g_k)_{x=x_0} / N(x_0) \qquad (3)$$

with

$$N(x_0) = \int_{x_0}^{\infty} (f_k f'_k + g_k g'_k)\mathrm{d}x. \qquad (4)$$

Using the normalisation

$$\int_0^{\infty} (f_k^2 + g_k^2)\mathrm{d}x = 1 \qquad (5)$$

we may put $N(x_0) = 1$ to a good approximation, since $x_0 \approx \alpha^2 Z A^{\frac{1}{3}} \ll 1$; the corresponding relative error in $N(x_0)$, and hence in $\delta\varepsilon$, being considerably less than x_0. Then †

$$\delta\varepsilon = 2a(g_k \delta f_k - f_k \delta g_k)_{x=x_0}. \qquad (6)$$

Denoting the regular and irregular Coulomb functions appropriate to a point charge Ze and an energy ε by $aF_k^{(\mathrm{r})}$, $aF_k^{(\mathrm{i})}$ and $G_k^{(\mathrm{r})}$, $G_k^{(\mathrm{i})}$ for the small and large components respectively, we may write for $x \geqq x_0$, where the potential is that due to a point charge,

† This derivation shows clearly that any contribution to $\delta\varepsilon$ arising from a change of the normalisation integral between isotopes is quite negligible compared to the dominant source of error to be mentioned subsequently. This is contrary to a contention of Reiner [8]), whose basic expression is not the exact one corresponding to (3).

$$f_k = a(C_1 F_k^{(r)} + C_2 F_k^{(i)}) \qquad (x \geqq x_0) \qquad (7)$$
$$g_k = C_1 G_k^{(r)} + C_2 G_k^{(i)}$$

and similarly for f'_k, g'_k. Thus C_2/C_1 gives the amount of admixture of the irregular Coulomb solution due to the deviation of the charge distribution from a point charge. Then using (7) to obtain δf_k and δg_k and substituting into (6), this becomes

$$\delta\varepsilon = -2a^2 C_1^2 \delta \left(\frac{C_2}{C_1}\right) (F_k^{(r)} G_k^{(i)} - F_k^{(i)} G_k^{(r)})_{x=x_0}. \qquad (8)$$

Putting $K_k = f_k/g_k$, we obtain from (7)

$$\frac{C_2}{C_1} = \left[\frac{aF_k^{(r)} - K_k G_k^{(r)}}{K_k G_k^{(i)} - aF_k^{(i)}}\right]_{x=x_0}. \qquad (9)$$

Then obtaining the difference of (9) between two isotopes, (8) becomes

$$\delta E = Q \left[\frac{F_k^{(r)} G_k^{(i)} - F_k^{(i)} G_k^{(r)}}{K_k G_k^{(i)} - aF_k^{(i)}}\right]^2_{x=x_0} \frac{a\delta K_k(x_0)}{2} \qquad (10)$$

where, with the normalisation (5), $Q = 4a^2 C_1^2 mc^2$, and with the normalisation

$$4\pi \int_0^\infty (f_k^2 + g_k^2) \mathrm{d}\gamma = 1, \quad Q = 8\pi Z e^2 C_1^2.$$

Eq. (10) is the most general expression for the I.S. and takes rigorous account of the distortion of the electron wave-function by the extended nuclear charge distribution. This is in contrast to the expression obtained by perturbation theory [9, 10]) where the Coulomb functions appropriate to a point charge are used as zero order wave-functions. It is important to observe that the nuclear charge distribution and its difference enter (10) only through $K_k(x_0)$ and $\delta K_k(x_0)$ respectively.

It is possible to simplify (10) by making some very good approximations (see e.g. I). Thus for x_0 corresponding to a distance of the order of the nuclear radius the potential energy is very much larger in absolute value than the binding energy and it is a very good approximation to neglect the latter in evaluating the functions F, G at the point x_0. These may then be expressed in terms of Bessel functions. Since $x_0 \ll 1$, it is in addition a very good approximation to retain only the lowest power of x_0 in the expansion of these and eq. (10) may then be shown to become

$$\delta E = Q \frac{1}{2\Gamma^2(2\sigma)} \frac{a\delta K_k(x_0)}{[a-(k+\sigma)K_k(x_0)]^2} \qquad (11)$$

where $\sigma = (k^2 - a^2)^{\frac{1}{2}}$ and Γ denotes the gamma-function. This expression was already given in I. Only the case of an s electron ($k = -1$) need be

considered, since this is the only practically important one, and the subscript k may therefore be dropped.

The quantity Q depends on the optical electron wave-function through $\psi^2(0)$ where $\psi(0)$ is the value of the Schrödinger wave-function at the origin, but there is also a correction arising from the inner electrons [1,11]). The determination of Q constitutes the principal uncertainty in extracting the nucleus-dependent part, i.e. $\delta E/Q$, from the I.S. data. A recent discussion of the determination of Q has been given by Breit [2]), who reaches the conclusion that one should not rely on the accuracy of such a determination to better than a few %. Such uncertainties are very much larger than any arising from the approximation made in obtaining (11) from (10) and also (6) from (3) (i.e. in putting $N(x_0) = 1$).

To obtain the I.S. from (11) for any particular charge distribution and its difference between isotopes it is necessary to know the corresponding values of $K(x_0)$ and $\delta K(x_0)$. It was shown in I that $K(x_0)$ and $\delta K(x_0)$ are determined very accurately by only $\langle r^2\rangle^{\frac{1}{2}}$ and $\delta\langle r^2\rangle^{\frac{1}{2}}$ respectively, where $\langle r^2\rangle^{\frac{1}{2}}$ is the root-mean-square radius of the nuclear charge distribution. If $R_u = \sqrt{\frac{5}{3}}\langle r^2\rangle^{\frac{1}{2}}$ is the radius of the equivalent uniform charge distribution and δR_u its difference, than as was shown in I, eq. (11) may be written in the form

$$\delta E = Q\,\frac{1}{2\Gamma^2(2\sigma)}\,\frac{a+(1+\sigma)K(a)}{a+(1-\sigma)K(a)}\,R_u^{2\sigma}\,\frac{\delta R_u}{R_u}. \tag{12}$$

The value of $K(a)$ is that appropriate for a uniform charge distribution; it is independent of R_u and is only a function of a, i.e. of Z, which was given in I. Thus, if R_u is obtained from the electron scattering results, the I.S. will directly give the value of δR_u. It is again to be emphasised that (12) is quite independent of the uncertainties attached to the perturbation theory expression for the I.S. and is as simple and more accurate than the latter. Also in contrast to corrected values of the perturbation theory expression for particular forms of charge distribution [8,11]), the result (12) is quite general and applicable to any form of charge distribution.

For comparison with the I.S. data it is useful to consider as a standard of reference the I.S. $(\delta E)_{std}$, evaluated using (12) with R_u obtained from the electron scattering data and with δR_u determined by the assumption that $R_u \propto A^{\frac{1}{3}}$. Then,

$$\frac{\delta E}{(\delta E)_{std}} = \frac{\delta R_u}{R_u} \Big/ \frac{\delta N}{3A} = \frac{3A}{R_u}\,\frac{\delta R_u}{\delta N} \tag{13}$$

where δN is the difference of the number of neutrons between the isotopes. Since the radius R_u which is used in $(\delta E)_{std}$ is the actual radius appropriate to the isotopes considered, $\delta E/(\delta E)_{std}$ is directly a measure of the corresponding fractional increase of radius $\delta R_u/R_u$.

3. Isotope Shift and Electron Scattering Results

The experiments on the elastic scattering of high energy electrons have given information not only about the overall extension of the nuclear charge distribution but also about its form. Thus, at present, the electron scattering data give a good representation of the individual nuclear charge distributions in terms of two parameters [12]). The thickness of the surface region is determined very accurately and is remarkably constant for all nuclei. Recently Elton [13]) has obtained a very good fit to the data with the nuclear density distribution $\rho(r) = \rho(0)[1+\exp(-c/d)]/[1+\exp((r-c)/d)]$ and making the assumptions: (1) the central density $\rho(0)$ is independent of A, (2) the surface thickness (and therefore d) is independent of A and (3) the forms of the neutron and proton distributions are the same, the latter thus being given by $(Z/A)\rho(r)$. Thus only c, the half-density radius, is assumed to depend on A. As a result of these assumptions Elton obtained expressions for R_u and c as functions of A, with $\rho(0)$ (or alternatively $\not{p}$, defined by $(\frac{4}{3}\pi)\rho(0)\not{p}^3 = 1$) and d as parameters. On fitting these expressions to the electron scattering data Elton obtained the values † $d = 0.540\pm0.015$ f (corresponding to a 90 %−10 % transition thickness of 2.38 ± 0.06 f) and $\not{p} = 1.115\pm0.005$ f. The expression for R_u corresponding to these values is

$$R_u = 1.115A^{\frac{1}{3}}+2.151A^{-\frac{1}{3}}-1.742A^{-1} \qquad (14)$$

which gives a very good fit to the electron scattering data for R_u. It should be noted that for the I.S. we are only interested in the charge distribution. Thus if eq. (14) is considered as only referring to this, it will be independent of the assumption about the equality of the forms of the neutron and proton distributions, although especially the high energy pion experiments [14]) indicate only a slight difference between these.

It is important to observe that, as may be seen from (14), R_u increases less rapidly than if it were proportional to $A^{\frac{1}{3}}$. In fact no fit can be obtained by assuming $R_u \propto A^{\frac{1}{3}}$. According to Elton this behaviour may thus be interpreted as a consequence of the constant thickness of the nuclear surface and of the assumed constancy of $\rho(0)$.

If we denote by $(\delta E)_{el}$ the value of the I.S. predicted by assuming that both R_u *and* δR_u are determined by (14), then

$$\frac{(\delta E)_{el}}{(\delta E)_{std}} = \frac{3A}{R_u}\frac{dR_u}{dA} = \frac{1-1.93A^{-\frac{2}{3}}+4.69A^{-4/3}}{1+1.93A^{-\frac{2}{3}}-1.56A^{-4/3}}. \qquad (15)$$

This is 0.86 for $A = 125$ and 0.90 for $A = 218$. As an appropriate average value we take the value of (15) for $A = 165$, which represents an appropriate mean value of A for those nuclei for which the I.S. is known:

† The length $f = 10^{-13}$ cm.

All te values given here and below take account of the corrections given by Elton [27]) to his original figures.

$$\frac{(\delta E)_{\mathrm{el}}}{(\delta E)_{\mathrm{std}}} = 0.88. \tag{16}$$

There is thus a reduction of the I.S. predicted by the electron scattering data relative to the value given by the assumption that $R_u \propto A^{\frac{1}{3}}$. This reduction is a consequence of the fact that the I.S. is determined by the difference of charge distribution only through δR_u and that this, as indicated by the electron scattering results, cannot be obtained by assuming $R_u \propto A^{\frac{1}{3}}$, as has previously been done for the I.S.

It should be noted, however, that eq. (14) and therefore more especially dR_u/dA (and hence eq. (15)) depend to some extent on the assumption of a constant $\rho(0)$ and thus cannot be considered entirely in the nature of an empirical fit to the electron scattering data. Thus, especially for heavier nuclei, which are the relevant ones for the I.S., the precise A-dependence of R_u will depend on an extrapolation from lighter nuclei, since for these $R_u/A^{\frac{1}{3}}$ shows the most pronounced decrease with A, and will therefore depend on the assumed constancy of $\rho(0)$ which will determine the precise form of this extrapolation. This assumption is intimately related to the effect of nuclear compressibility on the density distribution, and hence on the radius, of nuclei along the bottom of the mass valley. In the next section we shall give a tentative estimate of this effect and show that it is probably small, tending to increase slightly the values given by eq. (16). Perhaps future electron scattering results, leading to further improvements of knowledge about the nuclear charge distribution, will give more direct indications about this compressibility effect. For the present we may consider the values given by eq. (16) as the best consistent with present knowledge about the nuclear charge distribution.

In discussing the I.S. data it should be noted that eq. (16) is relevant to this only after allowance has been made for the effect on the I.S. data of differences of intrinsic deformation between isotopes [15, 4, 16]). The correlation between the I.S. and the variations of the intrinsic deformations is well established although there are perhaps some slight quantitative uncertainties. After allowing for this contribution to the I.S., using the intrinsic deformations obtained from Coulomb excitation, the I.S. data are consistent with[17])

$$\frac{(\delta E)_{\mathrm{vol}}}{(\delta E)_{\mathrm{std}}} = 0.65 \pm 0.1 \tag{17}$$

where $(\delta E)_{\mathrm{vol}}$ thus refers only to the 'volume' effect and where eq. (14) has been used for evaluating $(\delta E)_{\mathrm{std}}$, corresponding to a value of R_u given with sufficient accuracy by $R_u = 1.2A^{\frac{1}{3}}f$. The indicated error arises mainly from uncertainties in the determination of the electronic factor Q.

Thus there still remains a discrepancy between the I.S. data and the value (16) implied by the electron scattering data of a factor

$$\frac{(\delta E)_{\text{vol}}}{(\delta E)_{\text{el}}} = 0.74 \pm 0.1. \tag{18}$$

Although this is a considerable improvement over the previously accepted discrepancy (17), it must still be regarded, in spite of the rather large errors, as indicating a real difference between the I.S. data and the value predicted by the electron scattering data.

4. Isotope Shift and Nuclear Compressibility

The I.S. discrepancy was explained by Wilets *et al.*[4]) as due to nuclear compressibility. We shall follow these authors in ascribing the whole of the I.S. discrepancy to the effect of compressibility. This is perhaps justifiable as a first approximation since the I.S. discrepancy is an average quantity referring to many nuclei and may thus, in the same way as the nuclear extension, be considered primarily as a manifestation of the properties of nuclear matter. In any case it is of interest to investigate the consequences of this assumption. Of course for finer details the influence of shell structure and of the detailed properties of the wave-functions would have to be considered in any case. Thus it is possible that the odd-even staggering of the I.S. may be closely related to differences in the wave-functions of an odd and an even neutron.

Wilets *et al.* assumed that the compressibility affects the I.S. only through the Coulomb energy. They were then able to obtain a value for the compressibility coefficient of nuclear matter using the I.S. discrepancy (17). In extending their theory to include also the surface and symmetry energies we shall see, however, that the dependence of the symmetry energy on the density plays as important a role as the Coulomb energy, with the important consequence that the I.S. discrepancy cannot be uniquely related to the compressibility coefficient.

4.1. Effect of Compressibility on Changes of the Nuclear Radius

Following Wilets *et al.* we shall calculate changes of radius by considering the nucleus as a compressible liquid drop of uniform density with radius R. Subsequently (in 4.2 and 4.3) we shall relate the changes of radius estimated in this way with the I.S. discrepancy (18). The I.S. of such a liquid drop model of the nucleus will be determined by $\gamma_N = (3A/R)\partial R/\partial N$, which will be a measure of the reduction of the I.S. due to compressibility relative to the I.S. for incompressible nuclear matter for which $\gamma_N = 1$. Similarly we put $\gamma_Z = (3A/R)\partial R/\partial Z$ and $\gamma_R = (3A/R)\mathrm{d}R/\mathrm{d}A$. Also $\gamma_Z = \gamma_R = 1$ for incompressible nuclear matter and $\gamma_Z - 1$ and $\gamma_R - 1$ will be measures of the

effect of compressibility on changes of radius between isotones and between nuclei along the bottom of the mass valley respectively †.

The total energy is taken as the sum of the volume, Coulomb, surface, volume symmetry and surface symmetry energies:

$$E(\lambda) = E_0(\lambda)+E_C(\lambda)+E_s(\lambda)+E_\tau(\lambda)+E_{s\tau}(\lambda), \tag{19}$$

$\lambda = R_0/R$ is a scale parameter and the dependence on λ determines the dependence on the nuclear density; $\lambda \propto \rho^{\frac{1}{3}}$ and thus increasing λ corresponds to increasing ρ, $R_0 \propto A^{\frac{1}{3}}$ is the equilibrium radius when only E_0 is considered. Dashes will denote differentiation with respect to λ at $\lambda = 1$. Thus $E'_0 = 0$ and E''_0 is the compressibility coefficient of nuclear matter. The component energies of (19) are given by

$$E_0(\lambda) = u_0(\lambda)E_0(1), \quad E_C(\lambda) = \lambda E_C(1), \quad E_i(\lambda) = u_i(\lambda)E_i(1) \tag{20}$$

where i = s, τ or sτ, $u_0(1) = u_i(1) = 1$, and

$$\begin{aligned} E_0(1) &= -16.34\,A \text{ MeV}, \quad E_s(1) = 20.96\,A^{\frac{2}{3}} \text{ MeV} \\ E_C(1) &= 0.715\,Z^2A^{-\frac{1}{3}} \text{ MeV (corresponding to } R = 1.2A^{\frac{1}{3}}f) \\ E_\tau(1) &= 30.79(N-Z)^2/A \text{ MeV} \\ E_{s\tau}(1) &= -30.34(N-Z)^2/A^{4/3} \text{ MeV}. \end{aligned}$$

For the energies at $\lambda = 1$ we have taken the values obtained by Green [20]) from fitting the binding energies. Cameron [21]) has obtained a fit with somewhat different values, but the differences between these and Green's values are unimportant for our results. Green [22]) has also made a fit without $E_{s\tau}$ of which we need only note that $E_\tau(1)$ is reduced to $23.52(N-Z)^2/A$ MeV and that $E_C(1)$ is essentially unchanged ††. We have included $E_{s\tau}$ for the sake of generality, especially since the additional complications in estimating the effect of compressibility are only slight; $E_{s\tau}$ is presumably connected with a difference of the neutron and proton distributions in the surface. The coefficients $u(\lambda)$, which determine the dependence on the density, depend only on λ so that the dependence of $E(\lambda)$ on A, N and Z is completely

† Wilets *et al.* also considered a slight refinement of the simple liquid drop theory to allow for the effect of an increase of the density towards the surface arising from the Coulomb repulsion between the protons. Instead of taking the density as uniform throughout the nucleus they determined its distribution from minimisation of the total energy, subject however to the nuclear surface still being sharply defined and with the boundary condition that the radial derivative of the density is zero just inside the nuclear surface. This refinement, which takes no account of the diffuseness of the nuclear surface, made no appreciable difference to the results obtained for γ_N by the more primitive theory which simply assumes a uniform density. A similar conclusion for γ_R was reached by Feenberg [18]) and also by Swiatecki [19]).

†† In all these fits to the binding energies the dependence on the density, i.e. on λ, was ignored. Thus the numerical values obtained by such fits will not be those appropriate to exactly $\lambda = 1$. However, possible differences are expected to be rather small and would be quite unimportant for our results. A discussion of the effect of including the density dependence on the fit obtained to the binding energies has been given by Feenberg [23]).

determined by the corresponding dependence of the component energies for $\lambda = 1$.

Expanding $E(\lambda)$ about $\lambda = 1$ up to the second power in $(\lambda-1)$ and then minimising with respect to λ, the nuclear radius will be given by

$$\begin{aligned} R &= R_0\left(1-\frac{E_C+E'_s+E'_\tau+E'_{s\tau}}{E''_0+E''_s+E''_\tau+E''_{s\tau}}\right)^{-1} \\ &\approx R_0\left(1+\frac{E_C+k_sE_s+k_\tau E_\tau+k_{s\tau}E_{s\tau}}{KA+K_sE_s+K_\tau E_\tau+K_{s\tau}E_{s\tau}}\right). \end{aligned} \tag{21}$$

The density coefficients in (21) are independent of A, N or Z and are defined by $E''_0 = KA$, $E'_1 = k_1E_1$ and $E_1'' = K_1E_1$, where E_1 are the energies for $\lambda = 1$ given by (20).

Older values of K are between 50 and 100 MeV [23, 24, 19]). However, Brueckner and Gammel [5]), as the result of a recent calculation of the properties of nuclear matter, using nuclear forces consistent with the 2-body scattering data and having a hard core, obtain the considerably larger value $K = 167$ MeV. Wilets [6]) has obtained similar large values from a consideration of the surface energy and thickness using a semi-empirical statistical theory for the nuclear density distribution. Thus with the surface thickness as determined by the electron scattering experiments and with the value of E_s given in (20) Wilets obtains $K = 218$ MeV, while with Green's value $E_s = 17.97\,A^{\frac{2}{3}}$ MeV, corresponding to the omission of $E_{s\tau}$ in fitting the binding energies, Wilets obtains $K = 175$ MeV. In spite of the similar large values obtained from these two entirely different approaches it cannot be ruled out that these large values of K may be considerably in error.

Knowledge about the other density coefficients is very limited. For k_s Feenberg [23]) gave the very tentative estimate $-0.25 \leqq k_s \leqq 1$, based on permissible limits of the radii of mirror nuclei using the older, smaller, values of E''_0. Similar values, positive and of the order of unity, were obtained by Present [24]) as a result of calculations using the statistical method. With neglect of the symmetry energy terms, Wilets *et al.* gave an expression for R which was used by Reiner [8]) to estimate γ_N, in which $k_s = -2, K_s = 6$. This assumes $E_s(\lambda) \propto \lambda^{-2}$, i.e. that E_s is proportional to the surface area as the radius is changed for a given nucleus †.

There is, however, no justification for assuming that E_s varies in this way as the density changes. As an illustration only, we may consider the more plausible hypothesis that, for changes of radius of a given nucleus, E_s is proportional to the number of nucleons within a fixed distance from the nuclear surface, of the order of the nuclear force range. Then, if the latter is small compared to the radius, one obtains $k_s = 1$, $K_s = 0$, which is more in

† This assumption is justifiable for fission since for this the changes of surface area are assumed to arise only from changes in shape with the density remaining constant.

agreement with Present's and Feenberg's estimates. Rather tentatively we shall consider k_s to be positive and of the order of unity.

For k_τ, K_τ consideration of the kinetic energy of an ideal Fermi gas of free neutrons and protons gives $k_\tau = K_\tau = 2$. An early calculation by Present [24], including interactions to the first order, gave similar values. However, no recent calculations of k_τ, K_τ of the Brueckner type exist which would take account of 2-body correlations and use realistic nuclear forces, although Brueckner and Gammel [5]) have calculated $E_\tau(1)$ for which they obtain approximately 2.5 times the value for an ideal Fermi gas. This result indicates that k_τ, K_τ may possibly differ quite appreciably from the values for an ideal Fermi gas. No information exists about $k_{s\tau}, K_{s\tau}$. However, one may perhaps expect $k_{s\tau}$ to be of the same order as k_s.

From (21), remembering the A, Z and N dependence of the various terms, one obtains † on carrying out the differentiations and subsequently expanding in powers of $1/E''_0$

$$\gamma_N = 1-\frac{4E_C}{E''_0+E_C} -k_s\frac{E_s}{E''_0+E_C} +\bar{k}_\tau\frac{12Z}{N-Z}\frac{E_\tau}{E''_0+E_C} -\bar{K}_\tau\left(\frac{4Z-N}{N-Z}\right)\frac{4E_CE_\tau}{E''_0(E''_0+E_C)}, \tag{22}$$

$$\gamma_Z = 1+\frac{4E_C}{E''_0+E_C}\left(\frac{3}{2}\frac{A}{Z}-1\right) -k_s\frac{E_s}{E''_0+E_C} -\bar{k}_\tau\frac{12N}{N-Z}\frac{E_\tau}{E''_0+E_C} +\bar{K}_\tau\left(\frac{6A}{Z}\frac{2Z-N}{N-Z}+10\right)\frac{E_CE_\tau}{E''_0(E''_0+E_C)}, \tag{23}$$

$$\gamma_R = 1+\frac{4E_C}{E''_0+E_C}\left(\frac{3}{2}\frac{A}{Z}\frac{\mathrm{d}Z}{\mathrm{d}A}-1\right) -k_s\frac{E_s}{E''_0+E_C} +\bar{k}_\tau 12\left(\frac{Z}{N-Z}-\frac{A}{N-Z}\frac{\mathrm{d}Z}{\mathrm{d}A}\right)\frac{E_C}{E''_0+E_C} -\bar{K}_\tau\left(\frac{4Z-N}{N-Z}-\frac{1}{2}\frac{A}{Z}\frac{5Z-N}{N-Z}\frac{\mathrm{d}Z}{\mathrm{d}A}\right)\frac{E_CE_\tau}{E''_0(E''_0+E_C)}, \tag{24}$$

with

$$\bar{k}_\tau = k_\tau+\frac{E_{s\tau}}{E_\tau}k_{s\tau}, \quad \bar{K}_\tau = K_\tau+\frac{E_{s\tau}}{E_\tau}K_{s\tau}. \tag{25}$$

† Wilets *et al.* used a somewhat different procedure. Thus they put $R_0 \propto A^{\frac{1}{3}n}$ and determined n from the assumption that $R \propto A^{\frac{1}{3}}$ for nuclei along the bottom of the mass valley, i.e. that $\gamma_R = 1$. It is then easily seen that $n = \gamma_R - 1$ and that the resultant values γ'_N and γ'_Z are

$$\gamma'_N = \gamma_N-(\gamma_R-1), \quad \gamma'_Z = \gamma_Z-(\gamma_R-1)$$

where γ_N, γ_Z and γ_R are the values given by (22) – (24). Thus for γ_R close to unity the differences between γ'_N, γ'_Z and γ_N, γ_Z are small. A disadvantage of this procedure is that one needlessly sacrifices interesting knowledge about γ_R. Also this procedure cannot be considered as a consistent use of the liquid drop model since it assumes that nuclear matter does not saturate (since $n \neq 1$).

Of the terms with a higher power of E''_0 than the first in the denominator only the terms in K_τ and $K_{s\tau}$ with the second power of E''_0 in the denominator have been retained. All the other second and higher order terms are quite negligible for not too small values of E''_0. A small unimportant modification of the terms in $k_{s\tau}$ and $K_{s\tau}$ has also been made so that these may conveniently be grouped together with the terms in k_τ and K_τ respectively, corresponding to the expression (25). The quantity $\mathrm{d}Z/\mathrm{d}A$ is obtained from the relation [25]) $Z = A\ (1.981+0.01496\ A^{\frac{2}{3}})^{-1}$ appropriate to the stable nuclei along the bottom of the mass valley.

As expected the effects of nuclear compressibility decrease rapidly as E''_0 increases and for reasonably large E''_0 the $(\gamma-1)$ are very nearly inversely proportional to E''_0. The signs of the various terms in (22)—(24) are in agreement with elementary expectation; in particular the opposite signs, for positive k_τ, of the terms proportional to E_C and E_τ in γ_N and γ_Z should be noted. We also note the relation

$$\gamma_R = \gamma_N + \frac{\mathrm{d}Z}{\mathrm{d}A}(\gamma_Z - \gamma_N) \tag{26}$$

which is an immediate consequence of

$$\frac{\mathrm{d}R}{\mathrm{d}A} = \frac{\partial R}{\partial N}\frac{\mathrm{d}N}{\mathrm{d}A} + \frac{\partial R}{\partial Z}\frac{\mathrm{d}Z}{\mathrm{d}A}.$$

According to (22)—(24) the γ vary only slowly with A. The present errors of the I.S. data are in fact too large for the I.S. discrepancy obtained from this data to show any significant dependence on A. Numerical values used will therefore always be for $A = 165$, which is an appropriate average value of A for the I.S. data. The values of γ_N, γ_Z and γ_R for $A = 165$ are tabulated for several values of K (table 1).

TABLE 1

K	γ_N	γ_Z
50 MeV	$1-0.272-0.0715\,k_s+0.499\ k_\tau-0.03115\,K_\tau$	$1+0.715-0.0715\,k_s-0.715\,k_\tau+0.0383\ K_\tau$
100 MeV	$1-0.14\ -0.037\ k_s+0.2585\,k_\tau-0.0081\ K_\tau$	$1+0.37\ -0.037\ k_s-0.37\ k_\tau+0.0099\ K_\tau$
150 MeV	$1-0.094-0.025\ k_s+0.174\ k_\tau-0.0036\ K_\tau$	$1+0.25\ -0.025\ k_s-0.25\ k_\tau+0.00445\,K_\tau$
200 MeV	$1-0.071-0.019\ k_s+0.132\ k_\tau-0.00205\,K_\tau$	$1+0.189-0.019\ k_s-0.189\,k_\tau+0.0025\ K_\tau$

K	γ_R
50 MeV	$1+0.0855-0.0715\,k_s+0.062\ k_\tau-0.003\ K_\tau$
100 MeV	$1+0.044\ -0.037\ k_s+0.032\ k_\tau-0.00079\,K_\tau$
150 MeV	$1+0.03\ -0.025\ k_s+0.0215\,k_\tau-0.00035\,K_\tau$
200 MeV	$1+0.0225-0.019\ k_s+0.0162\,k_\tau-0.0002\ K_\tau$

There are several important comments to be made on the expressions (22)—(24) and the above numerical results. In contrast to γ_R which for

reasonable values of the density coefficients is close to unity, γ_N and γ_Z may differ considerably from unity. Also the Coulomb and the symmetry energy terms have a large effect on γ_N and γ_Z but only a small one on γ_R. These differences are due to the more rapid change of the Coulomb energy and especially the symmetry energy for variations of N and Z separately than for variations of both N and Z together appropriate to the stable nuclei along the bottom of the mass valley. Thus the reason why the symmetry energy has an effect on γ_N and γ_Z which is comparable to that of the Coulomb energy, although the latter is much larger, is that the symmetry energy varies much more rapidly between isotopes or isotones than the Coulomb energy, due to the corresponding rapid change of the neutron excess. On the other hand the effect of the symmetry energy on γ_R is small since in going along the mass valley the neutron excess changes only slowly. There is thus an amplification of the effect of compressibility on changes of radius between isotopes or isotones as compared with changes of radius between nuclei along the mass valley. The former changes can thus be expected to be a particularly good source of information about the density coefficients.

The contribution due to the Coulomb energy is considerably larger for γ_Z than for γ_N because the Coulomb energy changes more rapidly between isotones than isotopes. The surface energy makes the same contribution to γ_N, γ_Z and γ_R since it depends only on A. For values of k_s of the order of unity the contribution of this to γ_N and γ_Z is small compared to that due to the Coulomb and symmetry energies. Thus an important consequence is that the I.S. discrepancy cannot be uniquely related to E''_0 as Wilets *et al.* did, since the effect of the symmetry energy, through the unknown coefficients $\bar{k}_\tau$, $\bar{K}_\tau$, may be as important as that of the Coulomb energy.

4.2. DISCUSSION OF THE I.S. DISCREPANCY

The results of section 3 indicate that we should interpret (18) and not (17) as giving the reduction of the I.S. due to compressibility. Thus it is plausible to interpret γ_N as giving the reduction of the I.S. relative to $(\delta E)_{\mathrm{el}}$ instead of relative to $(\delta E)_{\mathrm{std}}$, i.e. we should put

$$\gamma_N = \frac{3A}{R_u}\frac{\partial R_u}{\partial N}\Big/\frac{3A}{R_u}\frac{\mathrm{d}R_u}{\mathrm{d}A} = 0.74 \text{ instead of } 0.65.$$

From (22), with the inclusion of only the Coulomb energy term, $\gamma_N = 0.65$ gives $K \approx 40$ MeV. This is essentially the value obtained by Wilets *et al.* and is much less than Brueckner and Gammel's value of 167 MeV which gives $\gamma_N = 0.915$ (only E_C). However, $\gamma_N = 0.74$ gives $K \approx 55$ MeV (only E_C). Although this is still much less than 167 MeV, the corresponding difference between $\gamma_N = 0.74$ and 0.915 could now to a large extent be covered merely

by the errors of the I.S. discrepancy. This would be true only to a much lesser extent with the previously accepted I.S. discrepancy (17), i.e. with $\gamma_N = 0.65$. If in addition we consider the possibility that the large values of E''_0 may be appreciably in error, as well as the limitations of the primitive theory used, then one could perhaps conclude that even with the inclusion of only the Coulomb energy there is no contradiction with the I.S. data. Thus for $K = 100$ MeV, $\gamma_N = 0.86$ and the difference between this and 0.74 could be almost completely covered by the errors of the I.S. discrepancy.

However, quite apart from any of these uncertainties, the above discussion is in any case quite inconclusive regarding the values of E''_0 deduced from the I.S. discrepancy because of the importance of the terms in k_τ, $\bar{K}_\tau$. On the other hand the effect of k_τ, $\bar{K}_\tau$ on γ_N is sufficiently large for one to use the I.S. discrepancy to obtain some information about these coefficients in spite of the uncertainties involved. For any assumed value of K (and k_s), eq. (22) will give a linear relation between those values of k_τ and $\bar{K}_\tau$ which are compatible with the I.S. discrepancy, i.e. with $\gamma_N = 0.74 \pm 0.1$. The precise value of k_s is unimportant, so long as this is of the order of unity, since the term in k_s will be dominated by the errors of the I.S. discrepancy. However, we shall take $k_s = 0.5$ as a plausible representative value. The resulting linear relations between k_τ and $\bar{K}_\tau$ for $A = 165$ and for $K = 50$, 100, 150 and 200 MeV are shown in the figure. In table 2 we have also given the corresponding maximum permissible values of k_τ (i.e. appropriate to $\gamma_N = 0.74 + 0.1$) for several $\bar{K}_\tau$.

TABLE 2

K \ $\bar{K}_\tau$	0	5	10	20	30
50 MeV	0.296	0.608	0.92	1.544	2.17
100 MeV	−0.006	0.15	0.306	0.617	0.929
150 MeV	−0.306	−0.202	−0.098	0.11	0.318
200 MeV	−0.603	−0.525	−0.448	−0.292	−0.136

The corresponding permissible values of k_τ, K_τ appropriate to Green's fit without the inclusion of $E_{s\tau}$ are simply obtained by replacing k_τ, $\bar{K}_\tau$ by $0.77\, k_\tau$, $0.77\, K_\tau$, i.e. by an upward parallel displacement of the lines in the figure such that their intersection with the ordinate axis is 1.3 times that shown. The corresponding maximum permissible values of k_τ are only slightly different from those given for k_τ in the above table. Then, remembering that for an ideal Fermi gas $k_\tau = K_\tau = 2$, the most surprising result is that for $K \gtrsim 100$ MeV, positive values of k_τ of the order of unity are only possible together with very large K_τ ($\gtrsim 30$), even if the errors are interpreted in the most favourable sense. Even for K as small as 50 MeV, $k_\tau \lesssim 0.5$ if

$K_\tau = O(1)$. Thus considering values of $K_\tau \gtrsim 10$ as physically unreasonable one is forced to negative values of k_τ, or at best small positive values, unless one is prepared to accept rather small values of $K(\lesssim 50$ MeV). If one would take the point of view that negative or very small positive values of k_τ are

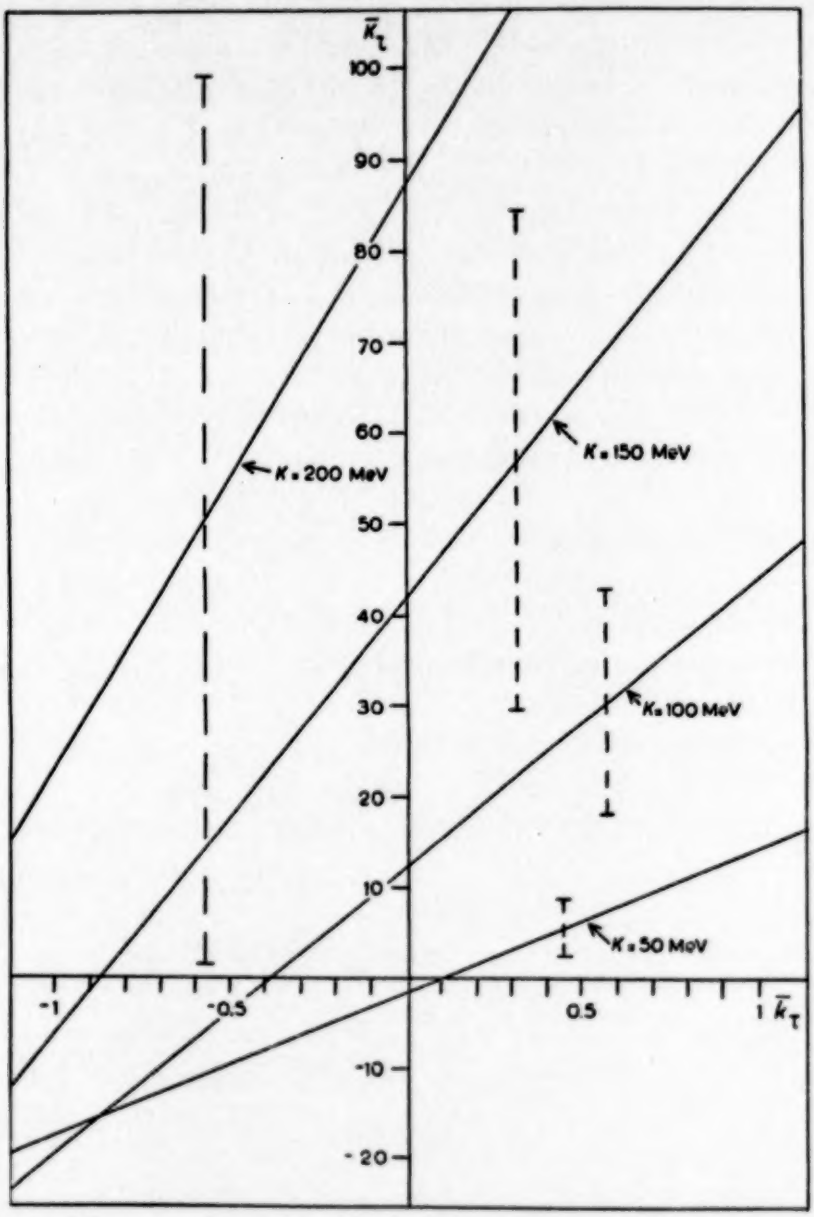

Fig. 1. The full lines show the relations between those values of $\bar{k}_\tau$ and $\bar{K}_\tau$ which are mutually compatible with the I.S. discrepancy $\gamma_N = 0.74$ for several indicated values of the compressibility coefficient, $E''_0 = KA$, and for $A = 165$. The dashed lines show the range of errors of these relations corresponding to an error of $\pm$ 0.1 in γ_N. The relations, corresponding to those in the figure, between the permissible values of k_τ and K_τ, appropriate to Green's fit without the inclusion of $E_{s\tau}$ are simply obtained by an upward parallel displacement of the lines in the figure such that their intersections with the ordinate axis is 1.3 times that shown.

not plausible, then, if one takes the results of the liquid drop model seriously, this situation could possibly be interpreted as indicating a value of K

considerably smaller than the large values recently obtained. It should be noted that with $\gamma_N = 0.65 \pm 0.1$ one would be forced to go to correspondingly even more negative values of k_τ than above. It would clearly be most interesting to know the results of a Brueckner type calculation for k_τ, K_τ.

With the inclusion of $E_{s\tau}$ we have from (25) that $\bar{k}_\tau = k_\tau - 0.216\ k_{s\tau}$ ($E_{s\tau}/E_\tau = -0.216$ for $A = 165$). Thus positive values of k_τ of the order of unity are now possible, but only together with rather large positive values of $k_{s\tau}$. Thus to have $k_\tau = 1$ together with $\bar{K}_\tau = O(1)$ it will be seen from the figure that we require $3 \lesssim k_{s\tau} \lesssim 5$ for $K = 50$ MeV, and $6 \lesssim k_{s\tau} \lesssim 11.5$ for $K = 150$ MeV. However, with $k_{s\tau} = O(1)$, as seems most plausible, the situation is essentially the same as above without the term in $E_{s\tau}$.

The most serious limitations of the primitive theory used is probably that it takes no proper account of surface effects. It is not clear whether the use of a more refined theory, such as the semi-empirical statistical theory of Wilets [6,26]), which takes account of the diffuseness of the nuclear surface, would change the above conclusions very much. However, the difficulties which these raise may possibly be interpreted as indicating that surface effects play a more vital role than can be represented merely by the surface and surface symmetry energy terms of the simple liquid drop model. It would clearly be desirable to use a theory of the semi-empirical statistical type in order to obtain a better understanding of the effects of compressibility on the I.S. An additional advantage of such a theory would be that the surface symmetry energy would be a consequence only of the dependence of the energy density on the proton and neutron densities and of the presence of Weizsäcker type gradient terms. As Wilets [6,26]) has shown, the magnitude of the latter could be determined from the surface energy and thickness. The volume and surface symmetry energies would then both be determined by the symmetry energy density, which would be specified by its value at the equilibrium density of nuclear matter and by its density dependence in the neighbourhood of this through k_τ, K_τ. Thus only this one set of density coefficients would enter, instead of the 2 sets required in the liquid drop theory. Thus not only could surface effects be better accounted for but also the ambiguity arising from the surface symmetry energy could in principle be resolved and more reliable knowledge about k_τ, K_τ might be obtainable from the I.S. data.

The possible importance of effects due to the diffuseness of the nuclear surface may be closely related to the fundamental question of to what extent it is justifiable to ignore shell structure effects, particularly the detailed properties of the wave-functions of those neutrons giving rise to the I.S. Thus the density distribution of neutrons filling shells of higher angular momentum will tend to be weighted towards the outside of the nucleus. As compared with neutrons having a more uniform density distribution such

neutrons, when added to a nucleus, will tend to penetrate less into the proton distribution and cause this to expand proportionally less with a correspondingly smaller I.S. The requirement of self-consistency of the neutron and proton distributions may be expected to reduce the importance of this effect somewhat. This type of shell structure effect, which is essentially a surface one, would thus be expected to work in favour of reducing the above difficulties met with in a straightforward application of the liquid drop model. It may be hoped that such shell structure effects would largely be accounted for in a self-consistent way by a statistical type theory which takes adequate account of the diffuse nuclear surface.

4.3. CHANGES OF RADIUS BETWEEN ISOTONES AND BETWEEN NUCLEI ALONG THE MASS VALLEY

Using (22) and (24), γ_R may be estimated rather directly from γ_N for any assumed value of K (and k_s), if we observe that the sum of the terms in k_τ and $\bar{K}_\tau$ in the expression for γ_R is very nearly proportional to the corresponding sum in the expression for γ_N. In this way one may obtain γ_R without needing to know k_τ, $\bar{K}_\tau$ and one may also hope to bypass some of the uncertainties of the theory. In table 3 we give the values of γ_R obtained from $\gamma_N = 0.74 \pm 0.1$ and also for comparison the values obtained directly from (24) by considering only the Coulomb energy term.

TABLE 3

K	γ_R^K from $\gamma_N = 0.74 \pm 0.1$	γ_R from eq. (24) with only E_c
50 MeV	$1.087 - 0.063\, k_s \pm 0.012$	1.0855
100 MeV	$1.0295 - 0.032\, k_s \pm 0.012$	1.044
150 MeV	$1.009 - 0.022\, k_s \pm 0.012$	1.030
200 MeV	$0.999 - 0.016\, k_s \pm 0.012$	1.0225

Thus for all except very small K, the values of γ_R obtained using the I.S. discrepancy are very close to unity if k_s is positive and of the order of unity as seems plausible. Also these values are somewhat closer to unity than those obtained by considering only the Coulomb energy, without making use of the I.S. discrepancy. Since the magnitude of $\gamma_R - 1$ will be a measure of the inadequacy of Elton's assumption of a constant central density it seems probable from the above results that this assumption is justified to a good degree of approximation and is consistent both with the electron scattering data and with the I.S. discrepancy, since small values of $\gamma_R - 1$ may very well be associated with quite large values of $1 - \gamma_N$ (and $\gamma_Z - 1$). Thus eq. (15), and hence (16), are probably justified to a good degree of approximation,

their modification due to compressibility being approximately obtained by multiplying by γ_R.

In the same way as for γ_R we may also obtain γ_Z from γ_N. The values of γ_Z obtained from $\gamma_N = 0.74 \pm 0.1$ and also those obtained directly from (23) by considering only the term in E_C are shown in table 4, together with the corresponding fractional increases of R_u between isotones. The latter will be given approximately by $(3A/R_u)\partial R_u/\partial Z = \gamma_Z (3A/R_u) \mathrm{d}R_u/\mathrm{d}A$ with $(3A/R_u)\,\mathrm{d}R_u/\mathrm{d}A = 0.88$ from (16).

TABLE 4

K	γ_Z	$\frac{3A}{R_u}\frac{\partial R_u}{\partial Z}$	γ_Z	$\frac{3A}{R_u}\frac{\partial R_u}{\partial Z}$
	from $\gamma_N = 0.74 \pm 0.1$		from eq. (23) with only E_C	
50 MeV	$1.698 - 0.174\, k_s \pm 0.15$	$1.495 - 0.153\, k_s \pm 0.13$	1.715	1.51
100 MeV	$1.542 - 0.090\, k_s \pm 0.15$	$1.358 - 0.079\, k_s \pm 0.13$	1.37	1.207
150 MeV	$1.486 - 0.061\, k_s \pm 0.15$	$1.308 - 0.053\, k_s \pm 0.13$	1.2495	1.1
200 MeV	$1.458 - 0.046\, k_s \pm 0.15$	$1.283 - 0.040\, k_s \pm 0.13$	1.188	1.045
	from eq. (26) with $\gamma_R = 1$ and $\gamma_N = 0.74 \pm 0.1$			
	1.46 ± 0.18	1.285 ± 0.16		

It should be noted that since γ_N, γ_Z and γ_R are related through eq. (26), γ_Z will be almost completely determined by only γ_N if γ_R is very close to unity (as will in any case be true for large K, as may be seen from table 3). If this is the case, then γ_Z will not give any more information about the density coefficients than that already given by γ_N. It will also be noted that, owing to the symmetry energy terms, the values of γ_Z obtained by considering only the term in E_C are very different from those obtained by using the I.S. discrepancy.

Thus, if the explanation of the I.S. discrepancy as a compressibility effect has any validity, a striking consequency should be that, on the average, the differences of R_u between isotones are appreciably *greater* than given by $R_u \propto A^{\frac{1}{3}}$. As for the I.S., allowance would have to be made for differences of intrinsic deformation between isotones. It would thus be most interesting to investigate changes of radius between isotones with electron scattering experiments. Apart from the intrinsic interest of such information this would provide a most interesting test of the effect of compressibility and may possibly also provide additional knowledge about the density coefficients to that given by the I.S. discrepancy.

5. Conclusion

A new derivation of the expression for the I.S. has been given. This derivation shows clearly the nature of the approximations made and that the uncertainties due to these are much less than any arising from the estimation of the electronic normalisation factor. The result (12) is as simple as the perturbation theory expression [9,10]) and is applicable to any form of charge distribution, while at the same time taking rigorous account of the distortion of the electron wave-function by this. The result shows that the I.S. depends in a very simple way on the charge distribution as a whole only through $R_u = \sqrt{\frac{5}{3}}\langle r^2\rangle^{\frac{1}{2}}$ and on the difference of charge distribution between isotopes only through the corresponding change δR_u.

Using this result the implications of the electron scattering results have been considered for the I.S., not only for its dependence on R_u *but also on* δR_u. Since the electron scattering data indicates that R_u increases less rapidly with A than if it were proportional to $A^{\frac{1}{3}}$, the I.S. predicted by these data is less than would be obtained for the same R_u but assuming δR_u to be determined by $R_u \propto A^{\frac{1}{3}}$. Elton's analysis [13]) of the electron scattering data, which assumes a constant central density, was used. The resulting reduction, which depends to some extent on this assumption, brings the I.S. predicted by the electron scattering data into better agreement with the 'volume' part of the I.S. data. However, there still remains a discrepancy of a factor $\approx 0.74 \pm 0.1$ between this and the predicted value.

Following Wilets, Hill and Ford [4]), the consequences have been investigated of interpreting the whole of this I.S. discrepancy as an effect of nuclear compressibility using a simple liquid drop model of the nucleus and including not only the Coulomb energy but also the surface energy and the symmetry energy. The dependence of the latter on the density turns out to have an effect as important as that of the Coulomb energy. Thus the I.S. discrepancy cannot be uniquely related to the compressibility coefficient of nuclear matter as Wilets *et al.* did. The surprising result is obtained that, with the large values of the compressibility coefficient ($\approx 150-200\,A$ MeV) indicated by recent investigations [5,6]), even the improved value of I.S. discrepancy obtained in this paper implies that the volume symmetry energy should decrease as the density increases or at best be almost stationary. This conclusion is subject to the condition that the surface symmetry energy does not increase unreasonably rapidly with the density. To obtain positive rates of increase of the order of that for an ideal Fermi gas a much smaller compressibility coefficient ($\lesssim 50\,A$ MeV) would then be required. Finally the implications of the I.S. discrepancy for changes of radius between nuclei along the mass valley and between isotopes have been considered. The effect of compressibility on the former changes is small and probably consistent with Elton's assumption of a constant central density. The differences of R_u

between isotones are predicted to be considerably greater than given by $R_n \propto A^{\frac{1}{3}}$. The liquid drop model used takes no adequate account of surface effects and thus the validity of the quantitative results obtained using this model will depend on the importance of effects due to the diffuseness of the nuclear surface. Such effects may also be expected to be closely related to shell structure effects. To obtain a better understanding of the effect of compressibility on the I.S. it would thus clearly be desirable to use a theory of the nucleon density distributions which accounts in a satisfactory way for surface effects. The semi-empirical statistical theory of Wilets should thus prove to be suitable.

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TERTIARY AND GENERAL-ORDER COLLISIONS

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Abstract: The general formalism of scattering theory is extended to cover reactions in which more than two particles exist in the entrance or exit channels, and the behaviour of the scattering matrix in this general case is discussed at arbitrary energies, in particular near thresholds. The energy dependence of observable quantities such as cross-sections and polarizations is given, and the occurrence and origin of the Wigner cusps and their counterpart in the general case are clearly seen; such singularities do not arise from the behaviour of the eigenphase-shifts.

Coulomb effects are not considered in the general case.

1. Introduction

The theory of many-channel reactions as it is usually formulated in quantum mechanics deals only with reactions of the type $a+X \rightarrow Y+b$, and does not consider cases when more than two particles are present in either ingoing or outgoing channels. In this paper the formalism is extended to cover the general case of any number of (neutral) particles in the external region, and the behaviour of the scattering matrix at thresholds is discussed. The results obtained are then applied to give the variation with energy of general scattering observables near thresholds; specific examples discussed are the n-particle differential cross-sections, the polarization, and the appearance of discontinuities in observables in old channels due to the opening of new channels. These last effects are seen to be a direct consequence of the orthogonality of the matrix which diagonalizes S.

This paper uses methods discussed in a previous paper on binary collisions [1]), which will henceforward be referred to as I, and to which the reader is referred for an explanation of the notation used.

2. Choice of Basis and Definition of the Scattering Matrix

Consider a general system undergoing scattering; this may be a two-particle scattering $a+X \rightarrow Y+b$, or it may involve more than two particles on the left or right hand sides, i.e., in the incoming and outgoing beams. We do not assume any restrictions as to the number of particles involved or the number of channels, except that they be finite; and of course they need not be elementary particles in any sense of the word. Then in the external

region the most general form of the wave function of the system is

$$\Psi^{\infty} = \sum_{\alpha} (A_{\alpha}\psi_{\alpha}^{\text{in}} - B_{\alpha}\psi_{\alpha}^{\text{out}}) \tag{1}$$

where $\psi_{\alpha}^{\text{in}}$, $\psi_{\alpha}^{\text{out}}$ are incoming and outgoing solutions of the wave equation in channel α in the external region (of course the channels are only defined in the external region). The scattering matrix $\mathbf{S}$ is then defined by

$$B = \mathbf{S}A; \tag{2}$$

$\mathbf{S}$ is unitary and can therefore be diagonalized by a real orthogonal matrix $\mathbf{T}$, in the form

$$\mathbf{S} = \mathbf{T}^{\mathrm{T}} \exp(2i\boldsymbol{\Delta})\mathbf{T} \tag{3}$$

with $\boldsymbol{\Delta}$ real.

This definition of $\mathbf{S}$ is quite general, and is exactly that used in the usual discussion of binary collisions (see I). $\mathbf{S}$ depends on the representation chosen, i.e., on the set α of channels used to expand Ψ^{∞}, and the problem of including tertiary and higher order collisions in the formalism is just that of finding a convenient set of observables to define the α. For two-particle channels, these are usually taken to be the total angular momentum J and its z-component M, the channel spin S, the channel orbital angular momentum l, and an index ρ to specify the two nuclei in the channel; when more than two particles are present in a channel, the choice of observables to define α is by no means unique, and we shall consider here one natural and convenient set; the methods used apply equally well to any other set.

Consider first channels containing three neutral particles. Calling these 1, 2, 3, we may choose as coordinates the vector $\mathbf{r}_{12}$ and the vector joining 3 to the centre of mass of 1 and 2, $\mathbf{r}_{31}+(m_2\mathbf{r}_{12}/(m_1+m_2))$, the centre of mass motion having been eliminated. We shall call these $\mathbf{r}_1$ and $\mathbf{r}_2$; they have corresponding reduced masses $\mu_1 = m_1m_2/(m_1+m_2)$, and $\mu_2 = \mu_1 m_3/(\mu_1+m_3)$. Then we can write down at once a wave-function representing an incoming or outgoing wave; it is

$$\psi_0^{\text{in, out}} = \psi_{l_1}^{\text{in, out}}(k_1r_1)\psi_{l_2}^{\text{in, out}}(k_2r_2)\xi_{123}\mathscr{Y}_{123} \tag{4}$$

where $\psi_l^{\text{in, out}}(kr)$ is the usual two-body asymptotic solution for angular momentum l:

$$\psi_l^{\text{in, out}}(kr) = \text{const} \times \frac{H_{l+\frac{1}{2}}^{2,1}(kr)}{(kr)^{\frac{1}{2}}}. \tag{5}$$

The case of charged particles is more complicated; the asymptotic form of the wave-function of three charged particles is not known, and we shall for this reason be unable to treat Coulomb effects; however, reactions involving only two charged particles in any channel, such as A(p, np)B, could be treated in this way.

$\mathscr{Y}$ is the angular part of the wave-function giving the dependence on

angle and spins, and ξ is the product of the intrinsic wave-functions of particles 1, 2 and 3.

Eq. (4) represents a solution with particles 1, 2 and 3 having sharp magnitudes of relative linear momenta k_1 and k_2, and is not very suitable as a basis for expanding the scattering matrix; it requires a continuum of channels to describe a physical state. A better one, which we shall use, is given by writing

$$\begin{aligned}(\mu_1/m)^{\frac{1}{2}}r_1 &= r\sin\phi, \quad (m/\mu_1)^{\frac{1}{2}}k_1 = k\sin\alpha,\\ (\mu_2/m)^{\frac{1}{2}}r_2 &= r\cos\phi, \quad (m/\mu_2)^{\frac{1}{2}}k_2 = k\cos\alpha,\end{aligned} \tag{6}$$

where m is a standard mass. We then have the convenient relation

$$k^2 = 2m(E-E_0)/\hbar^2 \tag{7}$$

where E is the total energy of the system and E_0 the internal energy of the channel considered, both in the C.M. system. Thus $E-E_0$ is the energy available as kinetic energy to the particles in the channel, in the C.M. system. With these coordinates the solutions have the form [2])

$$\begin{aligned}\psi^{\text{in, out}} = \text{const} \times \sin^{l_1}\phi\cos^{l_2}\phi\ {}_2F_1(-\lambda, l_1+l_2+\lambda+2; l_2+\tfrac{3}{2}; \sin^2\phi)\\ \times r^{-2}H^{2,1}_{l_1+l_2+2\lambda+2}(kr)\xi_{123}\mathcal{Y}_{123}\end{aligned} \tag{8}$$

where ${}_2F_1$ is a Jacobi polynomial and the new quantum number λ appears since we are now describing the scattering in terms of a single k. Note that a measurement of ϕ is physically equivalent to a measurement of α, since in the external region

$$\tan\phi = (\mu_1/\mu_2)^{\frac{1}{2}}r_1/r_2 = (\mu_1/\mu_2)^{\frac{1}{2}}v_1t/v_2t = \tan\alpha$$

and thus this wave-function describes the particles with a continuous distribution of energy between them, given by the amplitude

$$\sin^{l_1}(\alpha)\cos^{l_2}(\alpha)\ {}_2F_1(-\lambda, l_1+l_2+\lambda+2; l_2+\tfrac{3}{2}; \sin^2\alpha).$$

The set of functions (4) can of course be expanded in terms of (8); the relevant relation is [3])

$$\begin{aligned}&H_m(kr\cos\alpha\cos\phi)H_n(kr\sin\alpha\sin\phi)\\ &= \sum_{\lambda=0}^{\infty}(kr)^{-1}H_{m+n+2\lambda+1}(kr)\cos^m\phi\sin^n\phi\cos^m\alpha\sin^n\alpha\\ &\times {}_2F_1(-\lambda, m+n+\lambda+1; n+1; \sin^2\phi)\ {}_2F_1(-\lambda, m+n+\lambda+1; n+1; \sin^2\alpha)\\ &\times(-1)^\lambda 2(m+n+2\lambda+1)\frac{(m+n+\lambda)!(n+\lambda)!}{\lambda!(m+\lambda)!(n!)^2}.\end{aligned} \tag{9}$$

This relation gives the easiest way of extending the results to more than three particles. For given λ, the Jacobi polynomials are polynomials in the argument $\sin^2\alpha$; for convenience we give the first two

$$\begin{aligned}{}_2F_1(0, a; b; z) &= 1\\ {}_2F_1(1, a; b; z) &= 1-(a/b)z.\end{aligned} \tag{10}$$

The set (8) are orthogonal; the volume element in these coordinates is $r^5 \sin^2\phi \cos^2\phi \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\phi_1 d\phi_2 dr d\phi$.

It is often more convenient to work with the "regular" and "irregular" solutions $\mathfrak{F}_{(n)}$ and $\mathfrak{G}_{(n)}$, which can be defined by replacing H in (8) by J and N respectively, and are defined below for n-particle channels. In terms of these solutions we have (1) replaced by

$$\Psi^{\infty} = \sum_{\alpha}(A'_{\alpha}\mathfrak{F}_{\alpha} - B'_{\alpha}\mathfrak{G}_{\alpha})\xi_{\alpha}\mathscr{Y}_{\alpha}, \qquad B' = \mathsf{S}'\mathsf{A}', \qquad \mathsf{S}' = \mathsf{T}^{\mathrm{T}} \cot \Delta \mathsf{T}. \tag{11}$$

Note $\mathfrak{F}$ and $\mathfrak{G}$ then differ from the usual definition for two-particle channels by a factor r^{-1}. This form of the wave-function, with the radial part real, is more convenient for our purposes.

We can now see a systematic way of extending this basis to n-particle channels. We can write down a wave-function in the external region representing the n incoming or outgoing particles with sharp momenta; e.g. for incoming

$$\psi^{\mathrm{in}}_{0,n} = \psi_1{}^{\mathrm{in}}(k_1 r_1)\psi_2{}^{\mathrm{in}}(k_2 r_2) \ldots \psi^{\mathrm{in}}_{n-1}(k_{n-1} r_{n-1}) \tag{12}$$

using any orthogonal set of vectors $\mathbf{r}_1$ to $\mathbf{r}_{n-1}$ to define the positions of the n particles. These do not form a useful basis. We can then successively couple the two-body solutions $\psi_i{}^{\mathrm{in}}$ using (9) to get at each stage a series each term of which is a solution of the wave equation in the external region; and at each stage we lose a variable k_i and gain a quantum number λ_i. We are interested only in the behaviour of the radial part of the resulting function, and take

$$\psi_n{}^{\mathrm{in,out}} = \left(\frac{\pi m}{2\hbar}\right)^{\frac{1}{2}} H_p{}^{2,1}(kr) r^{\frac{1}{2}(5-3n)} \phi_n \mathscr{Y}_n \tag{13}$$

where we have suppressed all indices apart from the channel particle number n, and normalized to unit flux. $\phi_{(n)}$ is the product of the internal wave functions of the escaping nuclei in the channel, and $\mathscr{Y}_{(n)}$ contains all the spin and angle variables in the channel; both are normalized to one. p is given by

$$p = \sum_{i=2}^{n} l_i + 2\sum_{i=3}^{n} \lambda_i + \tfrac{1}{2}(3n-5). \tag{13a}$$

The regular and irregular n-particle solutions are then defined to have unit amplitude at infinity, and are given by

$$\mathfrak{F}_{(n)}(kr) = (\pi k/2)^{\frac{1}{2}} J_p(kr) r^{\frac{1}{2}(5-3n)}, \qquad \mathfrak{G}_n(kr) = (\pi k/2)^{\frac{1}{2}} N_p(kr) r^{\frac{1}{2}(5-3n)}. \tag{13b}$$

Note with this generalization to the n-particle case we have $k^2 = (2m/\hbar^2)(E - E_0)$ as for the two- and three-particle cases.

Using this set of functions, we can now write down the complete wave-function for the system in the external region for an eigenstate α of the scattering matrix. It is

$$\Psi_\alpha^{\infty} = A_\alpha \sum_\gamma (m_\gamma/\hbar k_\gamma)^{\frac{1}{2}} T_{\alpha\gamma}[\cot \delta_\alpha \mathfrak{F}_\gamma - \mathfrak{G}_\gamma]\phi_\gamma \mathscr{Y}_\gamma. \tag{14}$$

Nothe that this differs trivially from the form given in I for binary channels, owing to the different definition of $\mathfrak{F}$ and $\mathfrak{G}$ used there, but leads of course to the same T and $\boldsymbol{\Delta}$; A_α is a normalizing constant.

3. Behaviour near Channel Thresholds

We can now use (14), together with the behaviour of $\mathfrak{F}$ and $\mathfrak{G}$ for $k \to 0$, given by

$$\begin{aligned} \mathfrak{F}_{(n)} &\to \pi^{\frac{1}{2}} k^{p+\frac{1}{2}} 2^{-(p+\frac{1}{2})} (p!)^{-1} \gamma^{\frac{1}{2}(2p-3n+5)}, \\ \mathfrak{G}_{(n)} &\to -\pi^{-\frac{1}{2}} 2^{p-\frac{1}{2}} k^{-p+\frac{1}{2}} (p-1)!\, \gamma^{-\frac{1}{2}(p+3n-5)}, \end{aligned} \tag{15}$$

where p is defined by (13a), to find the behaviour of T and $\boldsymbol{\Delta}$ at channel thresholds or other energies. The method is exactly that used in I, appendix A, and the results are given in tables 1 and 2. The definition of k and the

TABLE 1

A_α and g_α

α	old	new
A_α	1	k^{p_α}
g_α	1	k^{2p_α}

TABLE 2

$g_{\alpha\beta}$

α \ β	old	new
old	1	k^{p_β}
new	k^{p_α}	$k^{\lvert p_\alpha - p_\beta \rvert}$

TABLE 3

$\Lambda_{\alpha\beta}$

α \ β		old	new
old	Λ^{R}	1	k^{p_β}
	Λ^{I}	1	k^{p_β}
new	Λ^{R}	k^{p_α}	$k^{(p_\alpha+p_\beta)}$
	Λ^{I}	k^{p_α}	$k^{(p_\alpha+p_\beta)}$

significance of $g_{\alpha\beta}$, g_α are as in I; that is to say, for a region round the energy E_0 considered

$$g_{\alpha\beta}T_{\alpha\beta} = \text{const}, \qquad g_\alpha \cot \delta_\beta = \text{const}, \tag{16}$$

while k is defined by $k^2 = (2m/\hbar^2)(E-E_0)$ for some arbitrary standard m.

These results can be used to give an effective range expansion of cot $\boldsymbol{\Delta}$ and T following the methods of I; the rapidity of convergence of such series for finite k is not known.

For convenience we give also in table 3 the corresponding variation with k of R at E_0, given by $\mathsf{R} = \mathsf{I}-\mathsf{S}$. The notation used is

$$R_{\alpha\beta}|_{E_0} = \text{const} \times \Lambda^{\mathsf{R}}_{\alpha\beta} + i \times \text{const} \times \Lambda^{\mathsf{I}}_{\alpha\beta}.$$

4. Behaviour of Observables near Channel Thresholds

The results of the previous section lead trivially to the energy dependence of observable quantities near channel thresholds.

4.1. REACTION CROSS-SECTIONS

The partial cross-section $\sigma_{\alpha\to\beta}$ has the energy dependence at energy E_0 given in table 4, where

$$\sigma_{\alpha\to\beta}|_{E_0} = \text{const} \times M_{\alpha\beta}.$$

TABLE 4

$M_{\alpha\beta}$

α \ β	old	new
old	1	k^{2p_β}
new	$k^{(2p_\alpha-1)}$	$k^{(p_\alpha+p_\beta-2)}$

These results are well known [4]) for $n = 2$; the cases $n = 3$ and general n with $\lambda_i = l_i = 0$ have been given by Guier and Hart [5]); note that their n is $(n-1)$ in our notation.

Table 4 implies that near threshold partial cross-sections corresponding to $\lambda_i = 0$ dominate. This means, if we consider a three-particle reaction such as A(n, 2n)B or A(p, 2n)B, that near threshold the outcoming neutrons will be distributed in momentum space as

$$\mathrm{d}\sigma(k_1, k_2) = \text{const} \times \sin^{2l_2+2}(\alpha) \cos^{2l_1+2}(\alpha)\, \mathrm{d}\alpha \tag{17a}$$

where $\alpha, l_1, l_2, \lambda_i$ are defined by eqs. (6), (4) and (8). Further, unless selection rules prohibit it, the dominant cross-section near threshold will be that for $l_1 = l_2 = 0$, and we will have

$$\mathrm{d}\sigma(k_1, k_2) = \text{const} \times \sin^2\alpha \cos^2\alpha \mathrm{d}\alpha \tag{17b}$$

which is also the distribution found by considering phase space arguments alone.

4.2. POLARIZATION

As a second example of the use of these results, we consider the polarization resulting from a nuclear reaction, near threshold; we limit ourselves to binary channels and consider the two cases of elastic scattering, for which the free channel energy does not change, and reactions for which it does; thus the two cases are new → new, and old → new or new → old.

The relative polarization in outgoing channel α' due to incoming channel α is given by an equation of the form [6])

$$\mathrm{d}p_{\alpha\alpha'} = \lambda_\alpha^2 \sum A_{\alpha\beta\gamma\alpha'} \operatorname{Im} [R^*_{\alpha\beta} R_{\gamma\alpha'}]/\mathrm{d}\sigma_{\alpha\alpha'}; \tag{18}$$

there is no contribution unless α, γ and β, α' refer to the same particles, owing to the orthogonality of the ϕ_γ in eq. (14). Near threshold the dominant terms are those referring to interference between $l = 0$ and $l = 1$ waves in the initial and final states; that is to say, writing reaction matrix elements in the abbreviated form $R^J_{l,l'}$ and suppressing other suffices, the dominant terms are those such as

$$\begin{aligned} R^{J*}_{0,0} R^{J'}_{1,1}&: \text{ scattering, or reaction without parity change;} \\ R^{J*}_{0,1} R^{J'}_{1,0}&: \text{ reaction with parity change.} \end{aligned} \tag{19}$$

The cross-section $\mathrm{d}\sigma_{\alpha\alpha'}$, however, behaves as an s-wave cross-section near threshold, whether or not a parity change takes place, and in the absence of other selection rules. Thus we have near thresholds

$$\begin{aligned} \mathrm{d}P_{\alpha\alpha'} &\propto k^2 \quad (\text{elastic scattering, } \alpha, \alpha' \text{ new}) \\ &\qquad\qquad\quad \alpha \text{ old, } \alpha' \text{ new,} \\ &\propto k \quad (\text{reaction,} \qquad\qquad \text{any parity}). \\ &\qquad\qquad\quad \alpha \text{ new, } \alpha' \text{ old} \end{aligned} \tag{20}$$

These results can obviously be extended to higher spin moments.

4.3. THE EFFECT OF THRESHOLDS ON OTHER CHANNELS

The opening of a neutral $l = 0$ two-particle channel causes cusps [5]) or "rounded steps" [7,8]) to appear in the cross-section vs. energy curves of other channels. The origin of these effects, with their generalization to higher angular momenta and many-particle channels, is readily seen from the results of the previous section. They arise, not from any singularity in the eigenphase-shifts, but from the orthogonality properties of T. Consider for simplicity an N-channel system in which at the energy considered one channel, say the N^{th}, is new. Then the dominant terms in an expansion of T in powers of k are, from table 3,

$$T_{\alpha\beta} = {}_0T_{\alpha\beta}(\alpha, \beta \neq N),$$
$$T_{\alpha N} = {}_0T_{\alpha N} k^{p_N}(\alpha \neq N), \tag{21}$$
$$T_{NN} = {}_0T_{NN},$$

and the relation

$$\sum_\gamma T^2_{\alpha\gamma} = 1 \tag{22}$$

implies that the $T_{\alpha\beta}$ contain a term ${}_1T_{\alpha\beta}k^{2p_N}$; this is reflected as a term proportional to k^{2p_N} in $\sigma_{\alpha\beta}$, and indeed in higher moments such as the polarization. This is the generalization of the usual result to N-particle channels and any angular momenta; p_N is given by equation (13). When more than one new channel opens at a given energy, each such channel γ contributes a term in $T_{\alpha\beta} \propto k^{2p_\gamma}$, and a term in observable quantities also $\propto k^{2p_\gamma}$. These terms were neglected in the effective range expansions of I.

For two-particle channels, $p_N = l_N + \frac{1}{2}$, so that for $l_N = 0$ all cross-sections $\sigma_{\alpha\beta}$ contain a term proportional to k. The constant of proportionality will in general be different above and below threshold [7]), leading to either a cusp or a rounded step in $\sigma_{\alpha\beta}$. Moreover, expanding (22), for $\beta \neq N$,

$$\sum_\gamma T^2_{\gamma\beta} = \sum_{\gamma\neq N} {}_0T^2_{\gamma\beta} + 2\sum_{\gamma\neq N} {}_0T_{\gamma\beta}\,{}_1T_{\gamma\beta}k^{2p_N} + {}_0T^2_{N\beta}k^{2p_N} + \cdots \tag{22a}$$

so that

$$2\sum_{\gamma\neq N} {}_0T_{\gamma\beta}\,{}_1T_{\gamma\beta} = -{}_0T^2_{N\beta}. \tag{22b}$$

This means that, if there is only one old channel γ, in which case $\beta = \gamma$ in (21b), then ${}_1T_{\gamma\gamma}$ and ${}_0T_{\gamma\gamma}$ have opposite signs, and the cross-section $\sigma_{\gamma\gamma}$ always *decreases* just above threshold. This was first shown in this case by Newton [7]). In the general case, however, there seems no reason why all the terms on the left hand side of (22b) should be negative.

The above considerations apply also to binary charged channels; in the notation of I, such new channels γ if repulsive reflect a term proportional to $\exp(-2\pi\eta_\gamma)$ in the other elements of T, so that all the derivatives with respect to k of $T_{\alpha\beta}$ are finite at $k = 0$, while if attractive they reflect a constant term, giving a finite step in the other $T_{\alpha\beta}$.

5. Summary

The formalism of many channel scattering theory has been extended here to cover channels containing more than two particles, and the behaviour of the scattering matrix has been investigated near thresholds. The methods used do not depend on the introduction of an interaction zone outside of which all forces vanish, but depend only on the assumed asymptotic forms of the wave-functions. These are valid provided the interparticle forces fall off sufficiently rapidly (faster than r^{-2}) at infinity. The behaviour of observable quantities follows simply from these results; in particular the

effect of thresholds in one channel upon observables in another is given in the general case.

I should like to thank Dr. R. J. Blin-Stoyle for helpful discussions of this work and the Shell Petroleum Company for the award of a Shell Scholarship.

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ON THE ENERGY DEPENDENCE OF THE ANISOTROPY OF HIGH-ENERGY NUCLEAR INTERACTIONS †

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Abstract: The energy dependence of the anisotropy parameter X defined for high-energy nuclear interactions has been determined in nucleon-nucleon and nucleon-nucleus collisions according to the Landau theory of meson production. An empirical formula $X = \left(0.82^{+0.06}_{-0.07}\right) \gamma_c^{(0.35 \pm 0.04)}$ is given for the variation of the degree of anisotropy with γ_c, the Lorentz factor of the CMS in the $10^{10} - 10^{14}$ eV energy range. It is shown on the basis of available experimental data that contrary to expectation no correlation is to be found between the number of shower particles n_s and the anisotropy, if $\gamma_c > 10$ and $n_s > 4$. An evaluation of the data on the basis of the "excited nucleon" model of meson production is given in the appendix.

1. Introduction

In the course of nuclear emulsion studies of multiple meson production several authors [1,2,3,4]) have observed anisotropic particle emission, in the centre of mass system (CMS), of colliding particles in nuclear interactions produced by high-energy cosmic ray particles. Experiments led to suppose anisotropy of the distribution to increase with increasing energy of the primary particles.

In order to obtain additional information on the mechanism of meson production, we determine in this paper the dependence of the degree of anisotropy on γ_c, the Lorentz factor of the CMS, by giving a formula for it in the $10^{10} - 10^{14}$ eV energy range. The problem of correlation between number of shower particles and anisotropy is also considered.

Experimental data are then compared to the value to be expected theoretically on the basis of the hydrodynamical model of meson production [5,6,7]) for nucleon-nucleon and nucleon-nucleus collisions. Furthermore the appendix contains an evaluation of the experimental data on the basis of the "excited nucleon" model of multiple production [8]).

The anisotropy parameter X introduced by Kaplon and Ritson [9]) will be considered as characteristic for the degree of anisotropy ††:

† This work was presented at the Conference on High-Energy Physics, Prague (June, 1958).

†† The use of this anisotropy parameter permits the maximum number of data to be obtained from the literature concerning angular distributions of high-energy nuclear interactions.

$$X = \frac{\vartheta_{\frac{1}{2}}/\vartheta_{\frac{1}{4}}}{(\vartheta_{\frac{1}{2}}/\vartheta_{\frac{1}{4}})_1}\,, \tag{1}$$

and if [10]) $\gamma_c \gg 1$,

$$X = \frac{\vartheta_{\frac{1}{2}}}{3\vartheta_{\frac{1}{4}}}\,;$$

$\vartheta_{\frac{1}{2}}/\vartheta_{\frac{1}{4}}$ represents the ratio of the angles at which $\frac{3}{4}$ and $\frac{1}{4}$ of all charged particles, respectively, are emitted in the laboratory system. The ratio $(\vartheta_{\frac{1}{2}}/\vartheta_{\frac{1}{4}})_1$ means the same for an angular distribution supposed to be isotropic in the CMS †. Thus X is equal to unity for isotropic distributions and it becomes > 1 for distributions which are anisotropic in the CMS.

2. Theoretical Considerations

Anisotropy in the CMS has been variously explained. According to Fermi [11,12]), in central collisions of high-energy nucleons, independently of energy, only isotropic particle emission can occur. Anisotropic particle emission is thus supposed to be due to non-central collisions of nucleons. The anisotropy is attributed here to the conservation of angular momentum. According to the model of "excited nucleon" [8]) anisotropy is similarly dependent on the impact parameter. In this theory, however, anisotropy also depends on the degree of nucleon excitation. (See appendix.)

According to the above theories high anisotropy can only be expected with low multiplicity (inelasticity < 1). On the other hand, rotational asymmetry with respect to the collision axis is required by these theories for non-central collisions. (Experiments, however, have not yet supported this assumption.)

According to Heisenberg [13]) anisotropy occurs as a result of the superposition of two effects. Partly it is supposed to be due to non-central collisions, partly to the fact that mesons interact strongly in the overlapping, high-temperature volumes of the Lorentz contracted nucleons.

In the hydrodynamical model of Landau [5,6]) only the latter effect is regarded to be the cause of the anisotropy. Because the Lorentz contracted volume decreases with increasing energy, according to the theories of Heisenberg and Landau, the anisotropy is expected to increase with increasing γ_c ††.

Although the theories of Heisenberg and Landau lead in this respect to the same result, since Heisenberg's theory concerning the nucleon-nucleus

† In the CMS we assume cylindrical symmetry with respect to the collision axis and mirror symmetry with respect to the equatorial plane of collision.

†† The factor γ_c^2 is proportional to the energy E_0 of the primary particle: $E_0 \approx 2\gamma_c^2$ in nucleon-nucleon collisions and $E_0 \approx 2\gamma_c^2 n$ in nucleon-nucleus collisions, where n is the number of nucleons struck in the nucleus ($Mc^2 = 1$).

collision is not yet consistently generalized, the dependence of the degree of anisotropy on γ_c according to eq. (1) was calculated only on the basis of the Landau theory.

Results were readily obtained using the integral angular distributions given in refs.[14,15]). The theoretical curve plotted for nucleon-nucleon collisions is shown in fig. 1, where the relation following from the hydrodynamical theory as modified by Fukuda *et al.*[7]) † is also plotted. They correspond to a power law, i.e.

$$X = c(n)\gamma_c^{\beta} \tag{2}$$

for $\gamma_c > 10$; corresponding values of $c(n)$ and β are given in table 2 for nucleon-nucleon and nucleon-nucleus collisions in accordance with the Landau and Fukuda theories, assuming the primary particle to collide on the average with $\bar{n} \approx 3.4$ nucleons in the nuclei of the emulsion [14]).

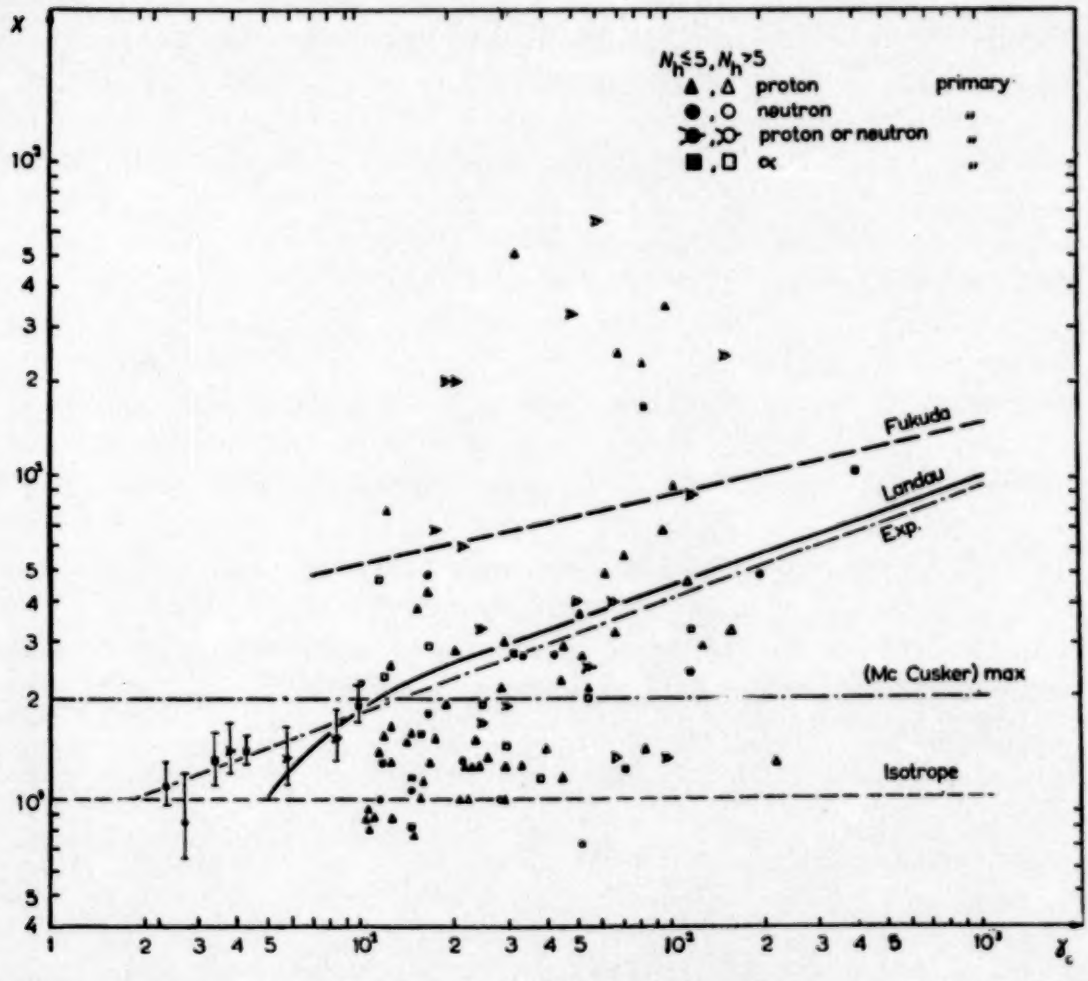

Fig. 1. Experimental values and theoretical curves of the anisotropy parameter X plotted versus the Lorentz factor of the CMS, γ_c.

† We express our thanks to Dr. M. Sato for sending us their paper before its publication.

In fig. 2, the parameter X plotted for $\gamma_c = 23.1$ is shown to decrease if the number of nucleons struck in the nucleus increases.

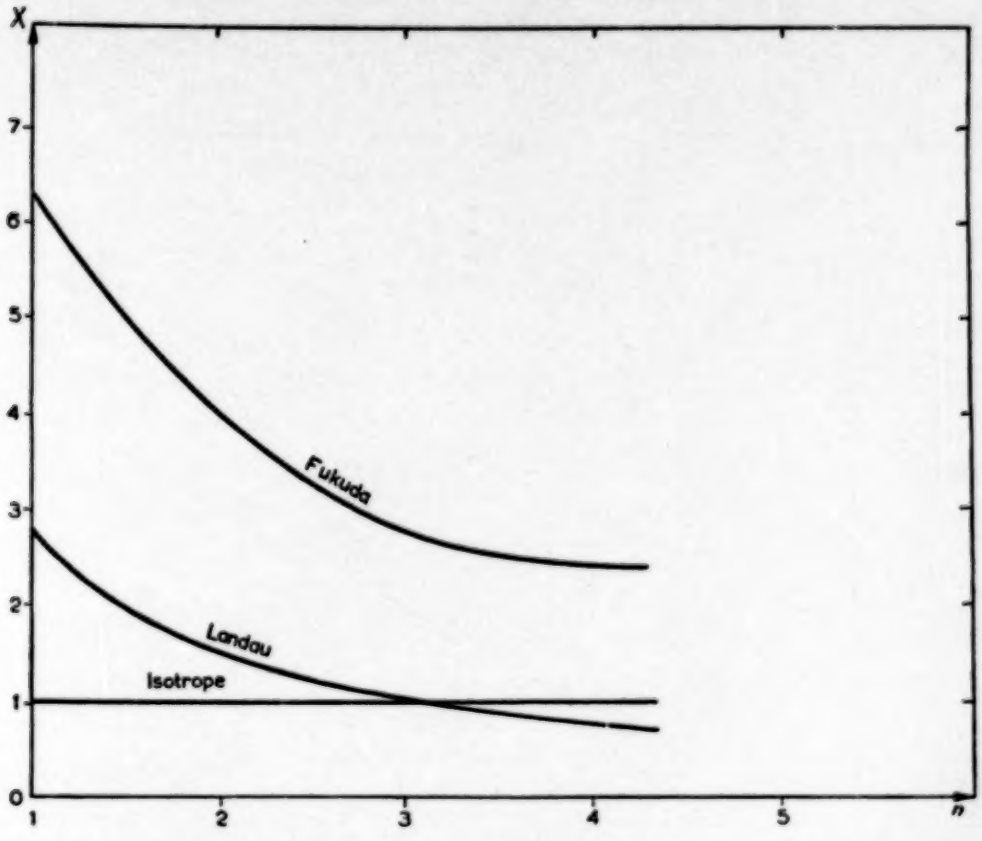

Fig. 2. Theoretical curves of X plotted versus n for $\gamma_c = 23.1$; n is the number of nucleons struck in the nucleus.

The upper limit of values obtained for X by Roesler and McCusker [10]) is also shown in fig. 1. This value has been obtained on the basis of a collision model similar to that corresponding to the modified Fermi theory by using the maximum impact parameter.

Since, according to the Landau theory, the number of charged particles emitted in the collision can be written as

$$n_s = 0.65(n+1)\gamma_c^{\frac{1}{4}} \tag{3}$$

and the anisotropy is also a function of γ_c as well as n, it is obvious that the anisotropy should be a function of n_s too. This function can also be expressed as a power law:

$$X = a n_s^{\delta} \tag{4}$$

where

$a = 0.76$ and $\delta = 0.68$ for the Landau theory,
$a = 2.58$ and $\delta = 0.46$ for the Fukuda theory,

in the case of nucleon-nucleon collisions. For nucleon-nucleus collisions, assuming the number of collisions to be $n \approx 3.4$, one has

$$a_L = 0.26, \quad a_F = 0.93.$$

The curves thus obtained are shown in fig. 3.

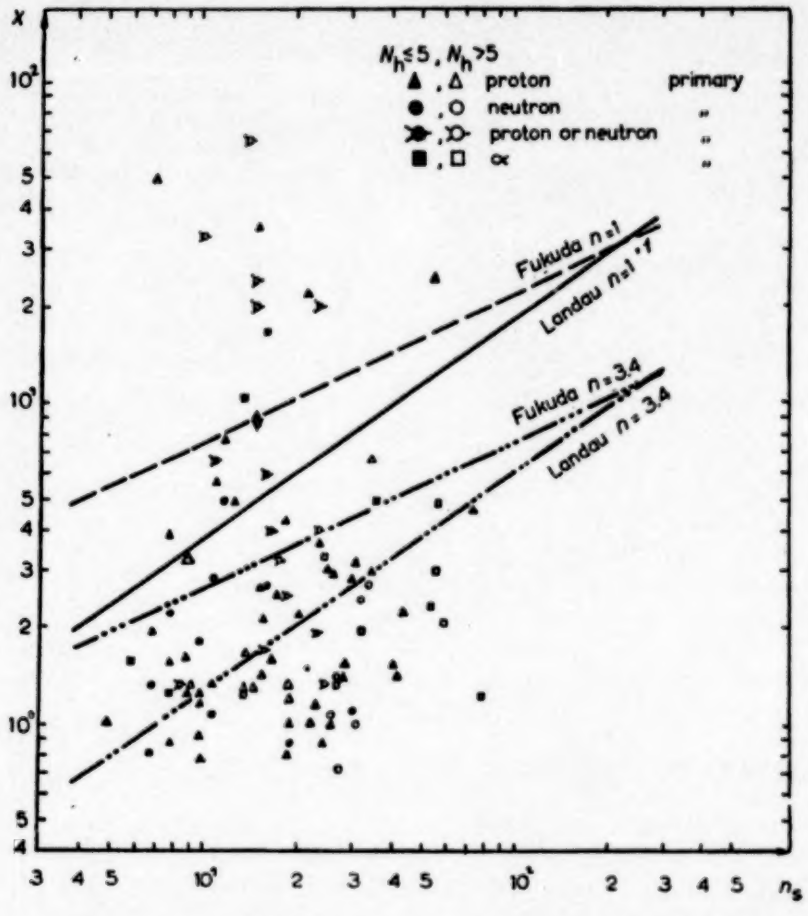

Fig. 3. Experimental values and theoretical curves of the anisotropy parameter X versus number of shower particles n_s.

3. Experimental Results

Nuclear interactions in which the number of shower particles was $n_s > 4$ were chosen to be evaluated according to eq. (1) [1–4,9,16–34]. The cases, in which the number of black and gray tracks was $N_h \leqq 5$ and $N_h > 5$ were treated separately; in the latter group also such cases were considered, in which the value of N_h was not indicated in the published papers. We applied a further selection according to the type of the primary particle.

The values thus calculated for X are plotted in terms of γ_c in fig. 1. Since a very great number of data of angular distributions was available in the energy range $\gamma_c < 10$ and no great fluctuation was observed in the anisotropy

parameters obtained, "composite stars" were constructed for which the average values $\bar{X}$ of the anisotropy parameter could be determined with greater accuracy. The mean value $\bar{\gamma}_c$ represents the weighted average of individual values γ_c.

TABLE 1

	Type of primary particle	$\bar{\gamma}_c$	Number of events	Total number of shower particles n_s	Average number of shower particles $\bar{n}_s$	$\bar{X}$	References
$N_h \leqq 5$	p	2.3±0.1	69	570	8.2±0.3	$1.1^{+0.2}_{-0.1}$	17, 18, 19)
	n	2.7±0.2	29	282	9.7±0,6	$0.85^{+0.3}_{-0.2}$	
	α	3.4±0.2	23	292	12.7±0.7	$1.3^{+0.3}_{-0.2}$	
	p or n	5.7±0.3	22	255	11.6±0.7	$1.33^{+0.3}_{-0.2}$	22)
	α	10.0±0.7	8	184	23.0±1.7	$1.9^{+0.3}_{-0.2}$	19, 21)
$N_h > 5$	p or n	3.8±0.2	14	239	17.1±1.1	$1.42^{+0.3}_{-0.2}$	20, 24)
	α	4.3±0.1	48	802	16.7±0.6	$1.41^{+0.15}_{-0.7}$	16, 20, 24)
	p or n	8.4±0.5	12	245	20.4±1.3	$1.53^{+0.3}_{-0.2}$	20, 23, 25)

It is seen from fig. 1 that the values of X show an increasing trend with increasing γ_c in spite of the rather large spread in values occurring for $\gamma_c > 10$. The possible cause for this spread will be discussed further below and in the appendix.

In order to obtain a quantitative relation for the degree of dependence of X on γ_c according to the above theoretical considerations, and assuming X to vary with γ_c according to a power law of the form $X = c\gamma_c^{\beta}$, a correlation calculation was carried out. Results of this calculation are given in table 2. In this table the constant values for c and β to be expected according to the hydrodynamical theory are also indicated.

It is seen from table 2 that the anisotropy parameter depends strongly on γ_c. There is no significant deviation between the two groups $N_h \leqq 5$ and $N_h > 5$.

As according to all theoretical considerations there must be a correlation between anisotropy and the number of shower particles, we also considered this problem. The slow increase in composite stars of the average anisotropy parameter $\bar{X}$ with increasing average number of particles $\bar{n}_s$ is illustrated in

fig. 4, while the values of X in terms of n_s with $\gamma_c > 10$ are shown in fig. 3.

In order to study the relationship between X and n_s a correlation calculation was again carried out, assuming a power law of the form $X = an_s^b$.

TABLE 2

			Corr. coeff. r	β	c
Experimental data	$N_h \leqq 5$		0.63 ± 0.1	0.31 ± 0.06	$0.83^{+0.08}_{-0.09}$
	$N_h > 5$		0.55 ± 0.08	0.37 ± 0.06	$0.82^{+0.08}_{-0.09}$
	Total		0.57 ± 0.06	0.35 ± 0.04	$0.82^{+0.06}_{-0.07}$
Theoretical values	according to Landau	$n = 1$		0.34	0.91
		$n = 3.4$		0.34	0.32
	according to Fukuda	$n = 1$		0.23	3
		$n = 3.4$		0.23	1.08

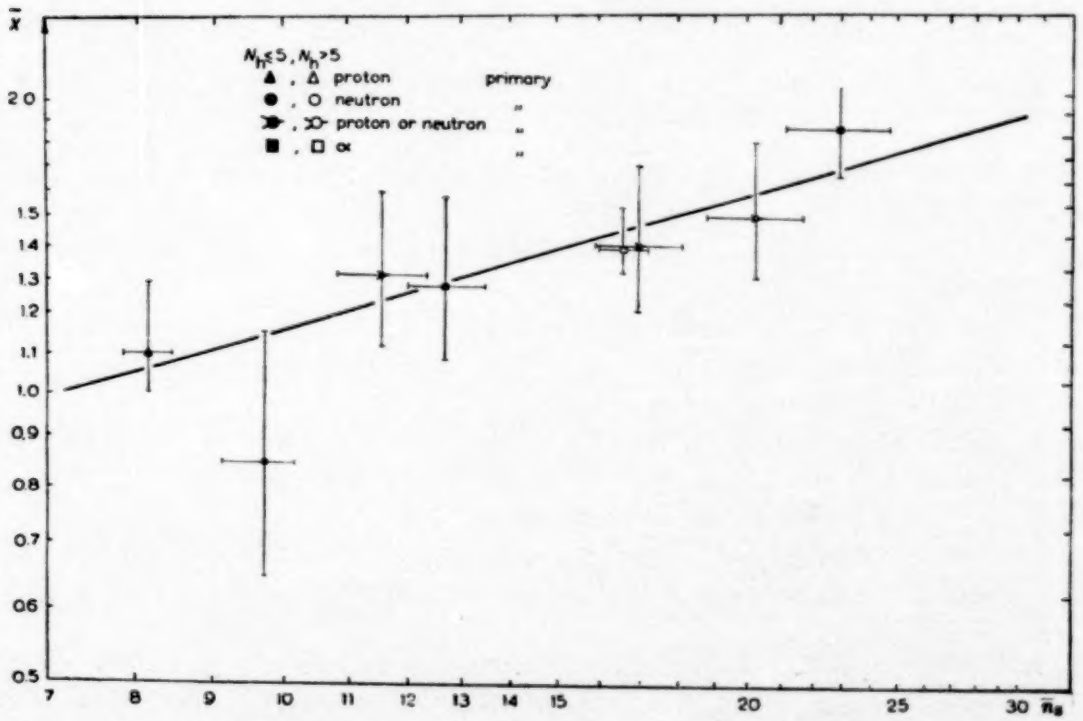

Fig. 4. Mean values $\bar{X}$ of X versus mean number $\bar{n}_s$ of shower particles n_s for $\gamma_c < 10$. The full line represents the curve best fitted by the experimental values.

The correlation coefficient for $\gamma_c < 10$ was found to be $r = 0.94 \pm 0.15$. The value δ was obtained as 0.47 ± 0.15 and that of a as 0.39 ± 0.20.
Thus we may say that according to our calculations the anisotropy increases slowly with increasing n_s if $\gamma_c < 10$. For $\gamma_c > 10$, however, the obtained correlation coefficient $r = 0.06$ indicates that in this range of energy the anisotropy is independent of n_s †.

4. Comparison of Theoretical and Experimental Results

a) Our result concerning the dependence of anisotropy on γ_c is in very good agreement with that expected for nucleon-nucleon collision according to the Landay theory. This is supported also by experimental data obtained in refs. [19]) and [20]) for various primary energy ranges (though these authors had a much smaller statistical material at their disposal).

b) For the average nucleon-nucleus collision a much lower value for $c(n)$ is predicted by the Landau theory than that obtained experimentally. The value to be expected according to the Fukuda theory is approximately the same as the experimental one. However, the value of the anisotropy exponent β predicted by the latter theory and the value experimentally determined do not agree too well.

c) According to both hydrodynamical models decreasing anisotropy is to be expected for nucleon-nucleus collisions. No change, however, has been found in the trend of the anisotropy parameter X for $N_h > 5$.

A similar result was obtained in ref. [20]) and according to these authors this constitutes a further argument against the validity of the hydrodynamical model.

d) The increase of anisotropy for $\gamma_c < 10$ seems to be in qualitative agreement with the theoretical considerations leading to the relation between the n_s and that in γ_c given in ref.[37]).

On the other hand, since no correlation has been found between anisotropy and n_s for $\gamma_c > 10$ it seems probable that no such relation as given by increase in eq. (3) holds between n_s and γ_c.

Our results seem to be in discrepancy with those obtained by Edwards *et al.*[23]) who have found increasing anisotropy with decreasing n_s. We must emphasize, however, that the latter authors selected jets according to the number of n_s in another way than that followed in the present paper.

e) The great fluctuation found in the values of X can be only partly explained by the statistical nature of the angular distribution (although considerable errors may be due to this fact). The observed spread may be attributed to the different impact parameters of collision, to the spread in

† *Note added in proof:* After having completed this work, we got new data on angular distributions of jets from the emulsion groups of Moscow and Berlin. These data have left our results unchanged within the limits of statistical error.

the number of nucleons struck in the nucleus and to an inaccurate determination of γ_c. On the other hand, the widely scattered values of the anisotropy parameters may be due, according to ref.[38]), to an interaction involving high impact parameters in a collision with a peripheric nucleon of the nucleus.

5. Conclusions

We may conclude from the study of anisotropy parameters that the theories explaining anisotropic emission by conservation of angular momentum only [10,12]) cannot explain the increase of anisotropy with increasing energy.

The dependence of the anisotropy on γ_c is in qualitative agreement with the theory of Landau. However, the fact that the anisotropy does not depend on n_s for $\gamma_c > 10$ shows that the relation (3) is not valid. From the latter fact one may conclude that the hydrodynamical theory in its present form cannot give either an unambiguous explanation for the mechanism of multiple meson production.

We are indebted to Dr. E. Fenyves for his valuable advice and we wish to express our thanks to the Emulsion Groups of Warsaw, Moscow, Berlin and Prague for sending us their data concerning angular distribution of jets prior to publication.

Appendix

The results given in our paper can be analysed on the basis of a model for multiple meson production proposed by Kraushaar and Marks [8]).

As we mentioned in § 2, according to this model the anisotropy depends not only on the impact parameter, but also on the degree of excitation of nucleons. The degree of excitation of nucleons is characterized by $\bar{\gamma} = (1-\bar{\beta}^2)^{-\frac{1}{2}}$, where $\bar{\beta}$ is the velocity of the excited nucleons in the CMS; the rest mass of the excited nucleon being $\gamma_c/\bar{\gamma}$. The value of $\bar{\gamma}$ has to be taken from the experimental angular distribution.

For the purpose of the analysis of the anisotropy on the basis of this model, it is necessary that the form of the connection between X and $\bar{\gamma}$ should be known. Integrating the angular distribution (1) given in ref.[8]) and transforming it to the laboratory system this connection is obtained as

$$\bar{\gamma} = \tfrac{1}{2}\left(3X+2-\frac{1}{X}\right)^{\frac{1}{2}}. \tag{5}$$

Relation (5) is plotted in fig. 5. Some experimental data are also shown in fig. 5. In these cases the experimental values of $\bar{\gamma}$ were obtained independently from X in the same way as described in ref.[39])

The requirements in selecting the events for analysis were that the two-cone structure should be marked, and the inequalities $N_h \leqq 5$, $n_s > 4$ should be

valid. (According to the latter requirements, probably most of the events were nucleon-nucleon interactions.) This analysis was carried out also for one event for which $n_s = 4$ (published in ref.[23]), and denoted by P_{20}) because it had an extremely high value of X.

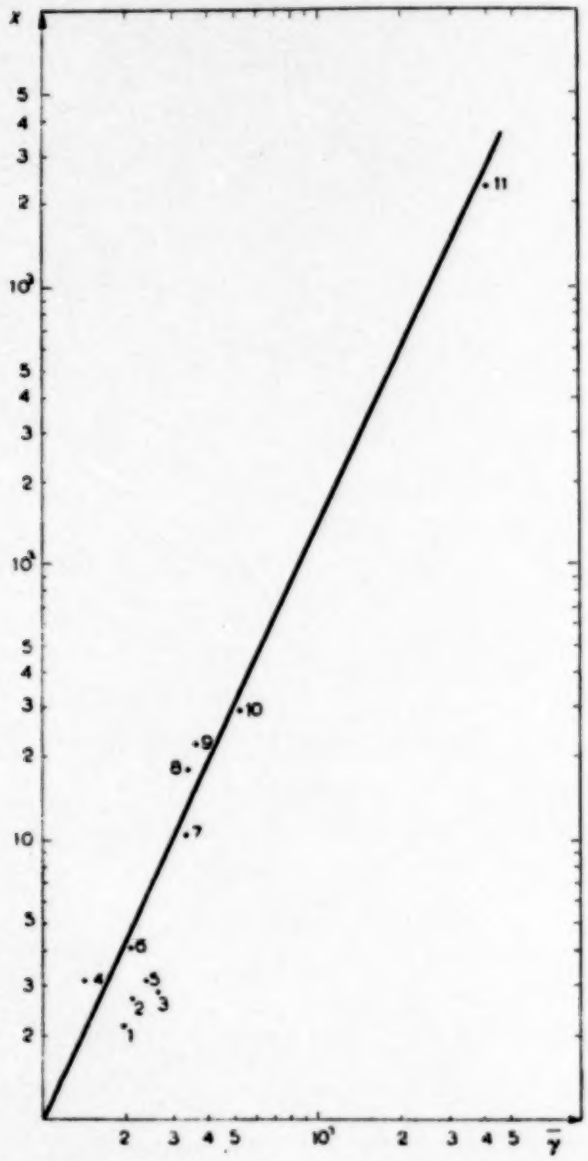

Fig. 5. Graphical representation of relation (5). The numbers indicate the events analysed: 1) No 18, ref.[39]); 2) ref. [31]); 3) ref.[29]); 4) P_5, ref.[23]); 5) No 26, ref.[39]); 6) No 155, ref.[39]); 7) No 115, ref.[39]); 8) ref.[4]); 9) P_2, ref.[22]); 10) ref.[1]); 11) P_{20}, ref.[23]).

As can be seen from fig. 5 the experimental points well fit the theoretical curve. This means that the value of $\bar{\gamma}$ can be calculated with the help of eq. (5) in a simple way by determining the anisotropy parameter X.

On the other hand one can determine the empirical dependence of $\bar{\gamma}$ on γ_c for this model with the help of eq. (5) and with the empirical formula

$X = c\gamma_c^{\beta}$, where the values of the constants c and β are given in table 2. The dependence of $\bar{\gamma}$ on γ_c is shown in fig. 6 (full line). In the same figure the mean values of $\bar{\gamma}$ are also plotted for different energy ranges taken from ref.[39]) (points). There is good agreement between our curve and the experimental data of ref.[39]). The last point has a rather large statistical error.

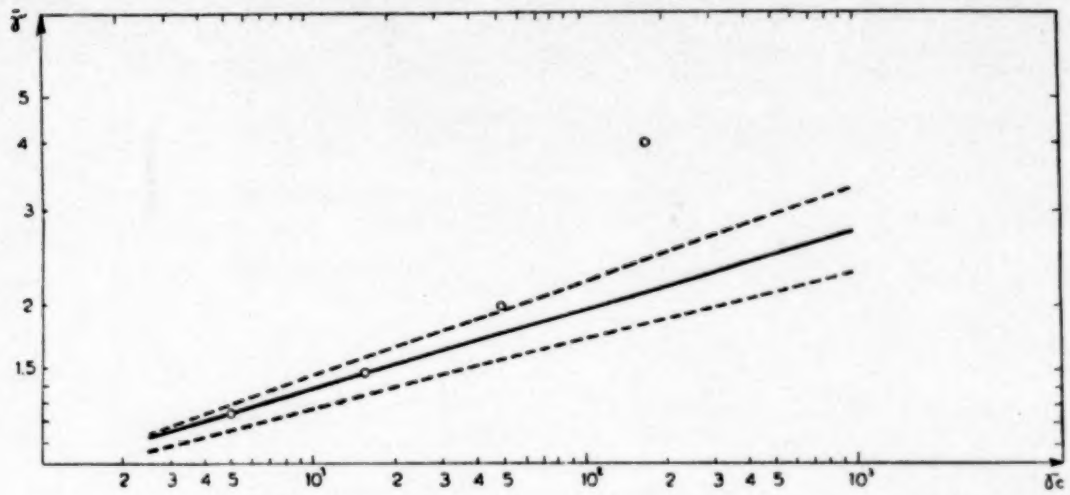

Fig. 6. Empirical dependence of $\bar{\gamma}$ on γ_c. The full line represents the result of our experimental analysis, the dotted line shows the statistical error. The points represent the experimental data of Ciok *et al.*[39]).

Fig. 6 shows $\bar{\gamma}$ to be an increasing function of γ_c. This fact means that with increasing γ_c an ever smaller fraction of γ_c will be available for exciting the nucleons †.

The great fluctuation found in the values of X may be explained — accepting this model — in a rather natural way by the fluctuation of the excitation parameter.

It must, however, be emphasized that in reality the conditions are more complicated than is assumed by this theory; for instance, the possible pion-pion interactions and the nucleon-nucleus collisions are not discussed.

† The exact theoretical form of $\bar{\gamma} = \bar{\gamma}(\gamma_c)$ would require a more complicated derivation from meson theory.

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ON THE POLARIZATION OF NUCLEONS IN HIGH ENERGY STRIPPING REACTIONS

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Abstract: Polarization of nucleons stripped from deuterons in high energy encounters with nuclei is determined on the basis of the generalized diffraction method in which account is taken of spin-orbit interaction.

1. Introduction

As is well known, interaction between nucleons and nuclei can be described by the optical model according to which the nucleus is phenomenologically considered as a body which can be characterized by some optical properties (refractive index and absorption coefficient). If the mean free path of the nucleon in nuclear matter is small compared with the dimensions of the nucleus the latter can be considered to be a black body. Absorption of nucleons by the nucleus perturbs the incoming wave and leads to elastic scattering which does not involve the formation of a compound nucleus. An important role in this case is played by spin orbit interaction which, as Fermi has shown [1]), leads to polarization of the scattered nucleons.

Besides absorption and elastic scattering which occur when nucleons interact with nuclei, some inelastic processes, such as stripping and diffractional disintegration may arise if more complex particles (deuterons) impinge on the nucleus.

In the stripping reaction one of the particles of the deuteron is captured by the nucleus whereas the second is liberated and carries off about half of the momentum of the incident deuteron. Spin-orbit interaction between the liberated particle and nucleus should lead in this case to polarization.

Interaction of nucleons or deuterons with nuclei can be investigated by the diffraction method [2]) which can be generalized to take into account spin-orbit coupling.

2. Diffractional Scattering of Neutrons

We shall start with the simplest problem on diffractional scattering of neutrons by absorbing nuclei. The wave length $\lambda\!\!\!^{-}$ of the incident neutron will be assumed to be small compared with the nuclear radius $R(\lambda\!\!\!^{-} \ll R)$. This is true for neutron energies exceeding 10 MeV.

Free motion of neutrons in a plane perpendicular to the wave vector of the incident neutron (z axis) can be described by the wave function

$$\psi_\varkappa = e^{i\varkappa\cdot\varrho}, \quad \int \psi_\varkappa^* \psi_{\varkappa'} d\varrho = \delta_{\varkappa\varkappa'},$$

where $\varkappa$ and ϱ are the projections of the neutron wave vector and radius-vector on a plane perpendicular to the z axis.

The wave function $\psi_0 = 1$ refers to incident neutrons. Owing to the presence of an absorbing nucleus the incoming wave is diffracted and the wave function of the neutron can be written in the form

$$\Psi = \psi_0 \Omega(\rho), \tag{1}$$

where in the case of an absolutely black nucleus the factor $\Omega(\rho)$ should be

$$\Omega(\rho) = \begin{cases} 0, \rho \leqq R \\ 1, \rho > R. \end{cases}$$

Part of wave function (1), namely,

$$\Psi' = \psi_0\{\Omega(\rho) - 1\}, \tag{2}$$

refers to scattered neutrons.

Expanding this part of the wave function into a series of functions $\psi_\varkappa$,

$$\Psi' = \sum_\varkappa a_\varkappa \psi_\varkappa,$$

and using the relation between the expansion coefficients $a_\varkappa$ and scattering amplitude [2]), $f(\vartheta) = -(ik/2\pi)a_\varkappa$, we obtain the following expression for the elastic scattering amplitude

$$f(\vartheta) = \frac{ik}{2\pi}\int \omega(\rho) e^{-i\varkappa\cdot\varrho} d\varrho, \tag{3}$$

where the scattering angle ϑ is related to $\varkappa$ by the equation $\varkappa = k\vartheta$ and $\omega(\rho) = 1 - \Omega(\rho)$.

For an absolute black nucleus the factor $\omega(\rho)$ in (3) is

$$\omega(\rho) = \begin{cases} 1, \rho \leqq R \\ 0, \rho > R. \end{cases} \tag{4}$$

If, however, the nucleus is considered as a semi-transparent body this factor should be [3])

$$\omega(\rho) = \begin{cases} 1 - \exp(-b\sqrt{R^2 - \rho^2}), & \rho \leqq R \\ 0, & \rho > R, \end{cases} \tag{5}$$

where b is the complex absorption coefficient. This choice of the factor $\omega(\rho)$ in (3) corresponds to an interaction between the neutron and nucleus which can be described by a complex potential $V(r)e^{i\delta}$.

Hitherto we have neglected the nucleon spin. The existence of a spin leads to appearance of an additional spin-orbit coupling between the nucleus and nucleon and account of this factor explains the polarization of nucleons scattered on nuclei. In the most general case the potential describing the

interaction between a nucleus and a nucleon with account of spin-orbit coupling can be written in the form

$$V = V(r)e^{i\delta} + \gamma e^{i\delta'} \frac{1}{r} \frac{dV(r)}{dr} \mathbf{l} \cdot \mathbf{S}, \tag{6}$$

where δ and δ' are phase shifts specifying the complex parts of the central and spin-orbit interaction; γ is the spin-orbit coupling constant, $\mathbf{l}$ is the nucleon orbital momentum and $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ is the nucleon spin (the nuclear spin is assumed to be zero).

The scattering amplitude which depends on the nucleon spin and corresponds to potential (6) can be represented in the form

$$f(\vartheta) = g(\vartheta) + \boldsymbol{\sigma} \cdot \mathbf{n} h(\vartheta), \tag{7}$$

where $\mathbf{n} = (\mathbf{k} \times \mathbf{k}')/|\mathbf{k} \times \mathbf{k}'|$ is a unit vector perpendicular to the nucleon scattering plane. Levintov [4]) (see also ref.[5]) and [6])) has shown that spin-orbit coupling can be taken into account by perturbation theory methods. The relation

$$h(\vartheta) = -ik^2 \vartheta \gamma e^{i(\delta' - \delta)} g(\vartheta) \tag{8}$$

then holds between the coefficients $h(\vartheta)$ and $g(\vartheta)$ which enter the expression for the scattering amplitude (7). This relation is independent of the central interaction.

If the optical model (5) is employed to describe the central interaction between the nucleon and nucleus one gets the following expression for the spin-independent part of the scattering amplitude:

$$g(\vartheta) = \frac{ik}{2\pi} \int \omega(\rho) e^{-i\boldsymbol{\varkappa} \cdot \boldsymbol{\rho}} d\boldsymbol{\varrho}.$$

Taking into account now relations (8) and (7) and using the evident equations

$$\mathbf{n} k^2 \vartheta = \mathbf{k} \times \boldsymbol{\varkappa} \quad \text{and} \quad \boldsymbol{\varkappa} e^{-i\boldsymbol{\varkappa} \cdot \boldsymbol{\rho}} = i \frac{\partial}{\partial \boldsymbol{\varrho}} e^{-i\boldsymbol{\varkappa} \cdot \boldsymbol{\rho}},$$

we obtain the following expression for the total scattering amplitude:

$$f(\vartheta) = \frac{ik}{2\pi} \int \omega(\rho) \left\{ 1 + \gamma e^{i(\delta' - \delta)} \boldsymbol{\sigma} \cdot \left(\mathbf{k} \times \frac{\partial}{\partial \boldsymbol{\varrho}} \right) \right\} e^{-i\boldsymbol{\varkappa} \cdot \boldsymbol{\rho}} d\boldsymbol{\varrho}. \tag{9}$$

Comparing this expression with expression (3) we find that account of spin-orbit coupling reduces to the substitution of the operator factor

$$\hat{\omega}(\boldsymbol{\varrho}) = \omega(\rho) \left\{ 1 + \gamma e^{i(\delta' - \delta)} \boldsymbol{\sigma} \cdot \left(\mathbf{k} \times \frac{\partial}{\partial \boldsymbol{\varrho}} \right) \right\} \tag{10}$$

for the factor $\omega(\rho)$ in the elastic scattering amplitude (3).

Expression (9) for the elastic scattering amplitude yields the well known

Fermi formula for nucleon polarization

$$P(\vartheta) = \frac{2\,\mathrm{Re}\,gh^*}{|g|^2+|h|^2} = \frac{\gamma \sin(\delta'-\delta)k^2\vartheta}{1+\gamma^2k^4\vartheta^2}. \tag{11}$$

Polarization of the nucleons as well as relation (8) do not depend on the nature of the central interaction.

3. Stripping Reaction at High Energies

We now employ the diffraction method to consider the stripping reaction at high energies of the order of 100 MeV. A theory of this reaction with account of finiteness of the nuclear radius can be developed if the nucleus is considered to be absolutely black [8]).

Let a fast deuteron with a wave vector $\mathbf{k}_d$ parallel to the z axis collide with the nucleus. For the sake of definiteness we shall consider a reaction in which a neutron is liberated and the proton is absorbed by the nucleus. This process can be described by the wave function

$$\Psi = \psi_0(\boldsymbol{\varrho}_d)\varphi_0(r)\Omega(\rho_n), \tag{12}$$

where $\psi_0(\boldsymbol{\varrho}_d) = 1$ is the wave function referring to the motion of the centre of mass of the deuteron in a plane perpendicular to the wave vector of the impinging deuteron, $\varphi_0(r) = \sqrt{(\alpha/2\pi)}/\mathrm{e}^{-\alpha r}/r$ is the deuteron ground state wave function, $\alpha = 1/2R_d$ and R_d is the deuteron radius. Evidently, the factor $\Omega(\rho_n) = 1-\omega(\rho_n)$ in (12) accounts for absorption of neutrons by the nucleus. If spin-orbit coupling is not taken into account this factor is defined by (4) for an absolute black nucleus.

By expanding Ψ into an integral of the functions $\psi_{\mathbf{k}}(\mathbf{r}_n) = (2\pi)^{-\frac{3}{2}}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_n}$ ($\mathbf{r}_n$ is the neutron radius vector) we find the probability amplitude $a_{\mathbf{k}}(\mathbf{r}_n)$ that the neutron wave vector will be $\mathbf{k}$ and the proton will be at a point $\mathbf{r}_p$

$$a_{\mathbf{k}}(\mathbf{r}_p) = (2\pi)^{-\frac{3}{2}}\int \varphi_0(r)\{1-\omega(\rho_n)\}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}_n}\mathrm{d}\mathbf{r}_n. \tag{13}$$

Integrating $|a_{\mathbf{k}}(\mathbf{r}_p)|^2$ over $\mathrm{d}\boldsymbol{\varrho}_p$ from $\rho_p = 0$ to $\rho_p = R$ we obtain the stripping differential cross section $\mathrm{d}\sigma(\mathbf{k})$ for the case when the wave vector of the liberated neutron is in the interval $\mathrm{d}\mathbf{k}$:

$$\mathrm{d}\sigma(\mathbf{k}) = \int_{\rho_p<R} \mathrm{d}\boldsymbol{\varrho}_p |a_{\mathbf{k}}(\boldsymbol{\varrho}_p)|^2 \mathrm{d}\mathbf{k}.$$

Making use of the fact that the functions $\psi_{\mathbf{k}}$ comprise a complete set, it is not difficult to find the integral cross section for the stripping reaction

$$\sigma_n = \pi R^2\left\{1-2\int_0^\infty \frac{p}{\xi}\,\mathrm{arctg}\,\frac{\xi}{p}\,\frac{j_1^2(\xi)}{\xi}\,\mathrm{d}\xi\right\},\quad p = \frac{R}{R_d}. \tag{14}$$

In the limiting case of $R \gg R_d$ this formula goes over to Serber's formula [7])

$$\sigma_n = \tfrac{1}{2}\pi R R_d,\quad R \gg R_d. \tag{15}$$

4. Effect of Spin-Orbit Coupling on Stripping

Consider now the effect of spin-orbit coupling on the stripping reaction. Since this coupling is small it only slightly changes the stripping integral cross section (14). However, account of spin-orbit coupling leads to a new qualitative result expressed by the fact that the stripped particles are polarized.

Spin-orbit interaction between the stripped neutron and nucleus can be taken into account by replacing $\omega(\rho_n)$ in (13) by the operator

$$\hat{\omega}(\varrho_n) = \omega(\rho_n)\left\{1+\gamma e^{i(\delta'-\delta)}\boldsymbol{\sigma}\cdot\left(\tfrac{1}{2}\mathbf{k}_d\times\frac{\partial}{\partial\varrho_n}\right)\right\}.$$

This substitution yields for the stripping probability amplitude the two-row matrix

$$a_{\mathbf{k}}(\mathbf{r}_p) = \int\varphi_0(r)\{1-\omega(\rho_n)\}e^{-i\mathbf{k}\cdot\mathbf{r}_n}\frac{d\mathbf{r}_n}{(2\pi)^{\frac{3}{2}}}$$
$$+\ \tfrac{1}{2}i\gamma e^{i(\delta'-\delta)}k_d\varkappa\boldsymbol{\sigma}\cdot\mathbf{n}\int\varphi_0(r)\omega(\rho_n)e^{-i\mathbf{k}\cdot\mathbf{r}_n}\frac{d\mathbf{r}_n}{(2\pi)^{\frac{3}{2}}}, \tag{16}$$

where $\mathbf{n} = (\mathbf{k}_d\times\boldsymbol{\varkappa})/|\mathbf{k}_d\times\boldsymbol{\varkappa}|$ is a unit vector perpendicular to the plane of the reaction.

Multiplying the matrix $a_{\mathbf{k}}(\mathbf{r}_p)$ by the Hermitian-conjugated matrix $a^+_{\mathbf{k}}(\mathbf{r}_p)$ and integrating the product over $d\varrho_p$ from $\rho_p = 0$ to $\rho_p = R$, we find for the stripping density matrix

$$\rho(\mathbf{k}) = \int_{\rho_p<R} d\varrho_p\, a_{\mathbf{k}}(\mathbf{r}_p)a^+_{\mathbf{k}}(\mathbf{r}_p). \tag{17}$$

Since in the following only the angular dependence of the cross section and polarization will interest us, the expression for the density matrix (17) should be integrated over all probable values of k_z:

$$\rho(\varkappa) = \int_{-\infty}^{\infty} dk_z \int_{\rho_p<R} d\varrho_p\, a_{\mathbf{k}}(\mathbf{r}_p)a^+_{\mathbf{k}}(\mathbf{r}_p). \tag{18}$$

Having determined the density matrix $\rho(\varkappa)$, we can find the differential cross section and polarization of the products of the stripping reaction by using the well known formulas

$$\sigma(\varkappa)d\varkappa = \tfrac{1}{2}\operatorname{tr}\rho(\varkappa)d\varkappa, \tag{19}$$

$$P(\varkappa) = \frac{\operatorname{tr}\{\boldsymbol{\sigma}\cdot\mathbf{n}\rho(\varkappa)\}}{\operatorname{tr}\rho(\varkappa)}. \tag{20}$$

The stripping differential cross section and polarization can be obtained in the limiting case $R \gg R_d$, when the boundary of the nucleus can be considered to be plane.

To evaluate the probability amplitude $a_{\mathbf{k}}(\mathbf{r}_p)$ in this case it will be convenient to expand the wave function of the ground state of the deuteron into a

Fourier integral

$$\varphi_0(\mathbf{r}) = \int \varphi(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}} d\mathbf{p}, \quad \varphi(\mathbf{p}) = \frac{1}{\pi(2\pi)^{\frac{1}{2}}} \cdot \frac{\sqrt{\alpha}}{\alpha^2+p^2}.$$

By introducing the new variables

$$\varrho'_n = \varrho_n - \mathbf{R}, \quad \varrho'_p = \varrho_p - \mathbf{R},$$

we can represent the probability amplitude (16) in the form

$$a_{\mathbf{k}}(\mathbf{r}_p) = (2\pi)^{-\frac{1}{2}} e^{-ik_z z_p - i\varkappa\cdot\mathbf{R}} \{(1-\tfrac{1}{2}i\gamma e^{i(\delta'-\delta)} k_d \varkappa\boldsymbol{\sigma}\cdot\mathbf{n}) \cdot \int d\pi\varphi(\pi, k_z) e^{-i\pi\rho'_p} \int_0^\infty dx'_n \int_{-\infty}^\infty dy'_n e^{-i(k_x-p_x)x'_n - i(k_y-p_y)y'_n} + \tfrac{1}{2}i\gamma e^{i(\delta'-\delta)} k_d \varkappa\boldsymbol{\sigma}\cdot\mathbf{n}(2\pi)^2 \varphi(\varkappa, k_z) e^{-i\varkappa\cdot\rho'_p}\}.$$

The x' axis is normal to the boundary of the nucleus. Performing the integration over x'_n, y'_n and p_y we get

$$a_{\mathbf{k}}(\mathbf{r}_p) = \sqrt{2\pi}\, e^{-i\mathbf{k}\cdot\mathbf{r}_p} \Big\{(1-\tfrac{1}{2}i\gamma e^{i(\delta'-\delta)} k_d \varkappa\boldsymbol{\sigma}\cdot\mathbf{n} \int_{-\infty}^\infty dp_x \varphi(p_x, k_y, k_z) \frac{e^{-i(p_x-k_x)x'_p}}{i(k_x-p_x)} + \tfrac{1}{2}i\gamma e^{i(\delta'-\delta)} k_d \varkappa\boldsymbol{\sigma}\cdot\mathbf{n} 2\pi\varphi(\mathbf{k})\Big\},$$

where in integrating over p_x the point $p_x = k_x$ should be passed from above. The remaining integration over p_x can be carried out by the method of residues. For the probability amplitude $a_{\mathbf{k}}(\mathbf{r}_p)$ the following expression is finally obtained

$$a_{\mathbf{k}}(\mathbf{r}_p) = \frac{\sqrt{\alpha}}{2\pi} e^{-i\mathbf{k}\cdot\mathbf{r}_p} \left\{\frac{e^{(P+ik_x)x'_p}}{P(P+ik_x)} - \tfrac{1}{2}i\gamma e^{i(\delta'-\delta)} k_d \varkappa\boldsymbol{\sigma}\cdot\mathbf{n}\left[\frac{e^{(P+ik_x)x'_p}}{P(P+ik_x)} - \frac{2}{\alpha^2+k^2}\right]\right\},$$

$$P = (\alpha^2+k_y^2+k_z^2)^{\frac{1}{2}}. \quad (21)$$

After substitution of (21) in the expression for the density matrix (18) the integration over x'_p should be performed from $-\infty$ to 0 and integration over y'_p should be replaced by integration over the perimeter of the nucleus

$$\rho(\kappa) = \int_{-\infty}^\infty dk_z \int_0^{2\pi} R\, d\varphi \int_{-\infty}^0 dx'_p\, a_{\mathbf{k}}(x'_p, \varphi) a_{\mathbf{k}}^+(x'_p, \varphi). \quad (22)$$

Here φ is the angle between vectors $\varkappa$ and $\mathbf{R}$ (vector $\mathbf{R}$ is directed along the x axis).

Using (22) and (21) we find, in accord with (19) and (20), for the cross section and polarization,

$$\sigma(\kappa) = \frac{\alpha R}{8\pi^2}\{I_1 + \tfrac{1}{4}\gamma^2 k_d^2 \kappa^2 (I_1 - 2I_2 + I_3)\},$$

$$P(\kappa) = \frac{\gamma \sin(\delta'-\delta) k_d \kappa (I_1 - I_2)}{I_1 + \tfrac{1}{4}\gamma^2 k_d^2 \kappa^2 (I_1 - 2I_2 + I_3)},$$

where the following notations have been introduced for the sake of brevity:

$$I_1 = \int_{-\infty}^{\infty} dk_z \int_0^{2\pi} d\varphi \frac{1}{P^3(\alpha^2+k^2)}, \quad I_2 = 4\int_{-\infty}^{\infty} dk_z \int_0^{2\pi} d\varphi \frac{P^2-k_x^2}{P(\alpha^2+k^2)^3}$$

$$I_3 = \int_{-\infty}^{\infty} dk_z \frac{8\pi R}{(\alpha^2+k^2)^2}. \tag{23}$$

Carrying out the integration in (23) we get

$$I_1 = \frac{4\pi}{\alpha^4(1+\zeta^2)^{\frac{1}{2}}}\left\{1-\frac{1}{2\zeta^3}[(1+\zeta^2)\operatorname{arctg}\zeta-\zeta]\right\},$$

$$I_2 = \frac{4\pi}{\alpha^4(1+\zeta^2)^{\frac{1}{2}}}\left\{\frac{3}{1+\zeta^2}-\frac{1}{2\zeta^3}[(1+\zeta^2)\operatorname{arctg}\zeta-\zeta]\right\},$$

$$I_3 = \frac{4\pi}{\alpha^4(1+\zeta^2)^{\frac{1}{2}}}\cdot\tfrac{1}{2}\pi p, \quad p = \frac{R}{R_d} \gg 1.$$

The dimensionless quantity $\zeta = \kappa/\alpha$ characterizes the angle of emission of the neutron.

Thus with account of spin-orbit coupling the angular distribution of neutrons produced in stripping is represented by the formula

$$d\sigma(\zeta) = \frac{RR_d}{\pi(1+\zeta^2)^{\frac{1}{2}}}\left\{1-\frac{1}{2\zeta^3}[(1+\zeta^2)\operatorname{arctg}\zeta-\zeta]\right.$$
$$\left.+\tfrac{1}{4}\gamma^2 k_d^2\alpha^2\zeta^2\left(1-\frac{6}{1+\zeta^2}+\frac{1}{2\zeta^3}[(1+\zeta^2)\operatorname{arctg}\zeta-\zeta]+\tfrac{1}{2}\pi p\right)\right\} d\Omega_\zeta, \tag{24}$$

where $d\Omega_\zeta = 2\pi\zeta d\zeta$. This formula is valid under the condition $\vartheta \ll 1$ (the emission angle ϑ being measured in radians), i.e., $\zeta \ll \sqrt{(E_d/\varepsilon)}$. If spin orbit coupling is neglected in (24) ($\gamma \to 0$) then Serber's formula for the angular distribution [7]) is obtained.

The polarization of neutrons stripped from deuterons at high energies is defined by the formula

$$P(\zeta) = \frac{\gamma \sin(\delta'-\delta)k_d\alpha\zeta(1-3/(1+\zeta^2))}{1-\frac{1}{2\zeta^3}[(1+\zeta^2)\operatorname{arctg}\zeta-\zeta]+\frac{1}{4}\gamma^2 k_d^2\alpha^2\zeta^2\left(1-\frac{6}{1+\zeta^2}+\frac{1}{2\zeta^3}[(1+\zeta^2)\operatorname{arctg}\zeta-\zeta]+\frac{1}{2}\pi p\right)}. \tag{25}$$

It follows from (25) that for an angle $\zeta = \sqrt{2}$ the neutron polarization changes sign.

Since Coulomb interaction was not taken into account it is obvious that formulas (24) and (25) will also define the cross section and polarization of stripped protons.

For small angles ($\zeta \ll 1$) expression (25) for the polarization is much simpler

$$P(\zeta) = -\frac{3\gamma \sin(\delta'-\delta)k_d\alpha\zeta}{1+\frac{3}{16}\pi\gamma^2 k_d^2\alpha^2 p\zeta^2} \quad \zeta \ll 1. \tag{26}$$

Comparison of this expression with (11) shows that in the small angle

region ($\zeta < \sqrt{2}$) the polarization of stripped neutrons is the opposite of that of neutrons scattered on nuclei.

In the small angle range maximum neutron polarization corresponds to the angle

$$\zeta_m = \frac{4}{\sqrt{3\pi p \gamma k_d \alpha}}. \tag{27}$$

This angle is $2/\sqrt{(3\pi p)}$ times smaller than the angle of maximal polarization which is observed when free neutrons are scattered on nuclei. In the small angle region the maximal polarization is

$$P_m = -\sqrt{\frac{12}{\pi p}} \sin(\delta' - \delta), \quad p = \frac{R}{R_d}. \tag{28}$$

Noting that even for the heaviest nuclei $p \approx 6$, we obtain for the maximal neutron polarization due to stripping a value which is of the same order of magnitude as that for scattering of free neutrons on nuclei.

Thus, owing to spin-orbit coupling neutrons and protons stripped from deuterons at high energies are strongly polarized. The peak in the angular distribution of the stripped protons and neutrons under this condition lies in the small angle region and hence the sign of polarization due to stripping is opposite to that due to scattering of free nucleons on nuclei.

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THEORY OF DIRAC PARTICLES WITH ORIENTED SPINS AND PARITY NON-CONSERVATION

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Abstract: The Lüders-Pauli theorem is investigated in connection with a type of theory chosen to describe Dirac particles with oriented spins.

1. Introduction

The discovery of parity non-conservation by Lee and Yang [1]) plays an important role in the theory of weak interactions.

It has been shown in a preceding note [2]) that Lee and Yang's results can be derived from the theory of Dirac particles with oriented spins.

Besides the Dirac equation, the free particle wave function ψ now satisfies the auxiliary condition [3,4])

$$s\psi = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}}{i\sqrt{(-\nabla^2)}}\,\psi \tag{1}$$

since the operator $\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}$ commutes with the Hamiltonian. The quantity $s = \pm 1$ characterizes the double spin projection on the direction of motion.

It has been remarked [2,5]) that in the case of β-decay or meson decay with neutrino emission summation over spin states should be performed not by the Casimir formula in which averaging is carried out over spin states, but by the following formula in which the spin states enter explicitly:

$$\begin{aligned}\langle\gamma',\gamma\rangle = {} & \tfrac{1}{16}\,\mathrm{tr}\,\gamma'\left(1+\rho_1\varepsilon' s'\frac{k'}{K'}+\rho_3\varepsilon'\frac{k'_0}{K'}\right)\\ & \times\left(1+s'\frac{\boldsymbol{\sigma}\cdot\mathbf{k}'}{k'}\right)\gamma\left(1+\rho_1\varepsilon s\,\frac{k}{K}+\rho_3\varepsilon\,\frac{k_0}{K}\right)\left(1+s\,\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{k}\right).\end{aligned} \tag{2}$$

Here ρ_μ, σ_μ are the well known Dirac matrices and $\rho_4 = \sigma_4 = I$; the matrices γ' and γ specify the type of interaction: scalar ($\gamma^{\mathrm{S}} = \rho_3$), pseudoscalar ($\gamma^{\mathrm{P}} = \rho_2$), vector ($\gamma^{\mathrm{V}} = I$), axial vector ($\gamma^{\mathrm{A}} = \boldsymbol{\sigma}$) or tensor ($\gamma^{\mathrm{T}} = \rho_3\boldsymbol{\sigma}$). The last three interactions have been written down for the non-relativistic case. The quantity $\varepsilon = \pm 1$ specifies the sign of the energy; the latter is related to the particle momentum $\mathbf{k}$ and rest mass k_0 by the equation $K = \sqrt{(k_0^2+k^2)}$.

Formula (2) can conveniently be used to study phenomena connected with longitudinally polarized electrons [6,7,8]). It may also be employed to describe polarization properties in weak interactions involving neutrinos. For this purpose the value of s should be fixed (the neutrino will be specified by unprimed letters K, k, $k_0 = 0$, whereas the primed quantities will refer to particles produced along with the the neutrino, e.g., an electron).

If for β^--decay one puts $\varepsilon' = 1$ for an electron and $\varepsilon = -1$ for an antineutrino, then, depending on the type of interaction, the matrix elements characterizing the spin states will be

$$\left.\begin{matrix} R^{S,T}_{s's} \\ R^{V,A}_{s's} \end{matrix}\right\} = \left.\begin{matrix} \rho_{33} \\ \rho_{44} \end{matrix}\right\} = 1 \pm ss'\beta \tag{3}$$

where $c\beta$ is the velocity of the emitted electrons and the values for ρ_{33} and ρ_{44} are given in ref.[4]) (see formulas (21.17) and (21.18)).

The definition of the neutrino and antineutrino requires some clarification and we shall therefore consider this point in the present paper. It seems most proper to use the term antineutrino for a particle which is created simultaneously with an electron in β^--decay. If the experimental results of Frauenfelder *et al.*[9]) are taken into account (according to these data electrons are emitted in β^--decay predominantly with $s' = -1$), then, to be consistent with (3) one must put, in the case of S, T coupling, $s = -1$ (left screw, according to Lee and Yang's definition) for an antineutrino and $s = 1$ (right screw, Feynman and Gell-Mann [10])) in the case of V, A coupling. For a combination of these types of interaction it may be necessary to introduce two different types of antineutrinos.

2. Lüders-Pauli Theorem in the New Theory of the Neutrino

The spinor amplitude for the free Dirac equation has the form [11])

$$b_{s\varepsilon}(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} f(\varepsilon K)\cos\theta_s \\ f(\varepsilon K)\sin\theta_s e^{i\varphi} \\ s\varepsilon f(-\varepsilon K)\cos\theta_s \\ s\varepsilon f(-\varepsilon K)\sin\theta_s e^{i\varphi} \end{pmatrix}, \tag{4}$$

where

$$\theta_s = \tfrac{1}{2}\theta - \tfrac{1}{4}\pi(1-s), \qquad f(\varepsilon K) = \sqrt{1+(k_0/\varepsilon K)},$$

and θ, φ are spherical angles of vector $\mathbf{k}$.

For a neutrino ($k_0 = 0$) the wave functions will be related by the equation [2])

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \tag{5}$$

where

$$\lambda = \varepsilon s_\varepsilon. \tag{6}$$

It follows from here that λ can assume four values which specify the spin direction for positive ($\varepsilon = 1$) and negative ($\varepsilon = -1$) energy values. If s_+ and s_- are respectively the values of s for $\varepsilon = 1$ and $\varepsilon = -1$, we may have

(a) $\lambda = \quad 1$: then $s_+ = -s_- = 1$,
(b) $\lambda = -1$: then $s_+ = -s_- = -1$,
(c) $\lambda = \quad \varepsilon$: then $s_+ = \quad s_- = 1$,
(d) $\lambda = -\varepsilon$: then $s_+ = \quad s_- = -1$.

Since the Dirac equation satisfies all relativistic invariance requirements it will be sufficient for us to confine ourselves to an investigation of the invariance of the auxiliary condition (5).

Under space inversion (transformation from right to left handed coordinate system) the wave function transforms according to $\psi' = \rho_3 \psi$:

$$\begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \psi'_3 \\ \psi'_4 \end{pmatrix} = -\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{7}$$

Moreover, in this case $\varepsilon' = \varepsilon$, $\mathbf{k}' = -\mathbf{k}$ (polar vector), $\mathbf{s}' = \mathbf{s}$ (axial vector), where $\mathbf{s}$ is a unit spin vector connected with s by the relation

$$s = \frac{\mathbf{s} \cdot \mathbf{k}}{K}. \tag{8}$$

Hence $s' = -s$, $\lambda' = -\lambda$ and therefore in the primed coordinate system the auxiliary condition (5) retains its invariant form ($I = \text{const}$) †.

For time reversal we have $\psi' = \rho_2 \psi$:

$$\begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix} = -i \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \begin{pmatrix} \psi'_3 \\ \psi'_4 \end{pmatrix} = i \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{9}$$

In this case $\varepsilon' = -\varepsilon$, $\mathbf{k}'_\mp = -\mathbf{k}_\pm$, $\mathbf{s}'_\pm = -\mathbf{s}_\mp$, $s'_\pm = s_\mp$, where $\mathbf{k}_+$ and $\mathbf{k}_-$ are the particle momenta when ε is equal respectively to $+1$ and -1.

Closely related to time reversal is the charge conjugation transformation considered in detail by Pauli [12])

$$\psi' = -i\alpha_2 \rho_3 \psi^{+\mathrm{T}}$$

or

$$\begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix} = i\sigma'_2 \begin{pmatrix} \psi_3^+ \\ \psi_4^+ \end{pmatrix}, \quad \begin{pmatrix} \psi'_3 \\ \psi'_4 \end{pmatrix} = -i\sigma'_2 \begin{pmatrix} \psi_1^+ \\ \psi_2^+ \end{pmatrix}, \tag{10}$$

where $\sigma'_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ is a two-row Pauli matrix and $\psi^{+\mathrm{T}}$ is a transposed matrix.

It can easily be shown in this case that

$$\varepsilon' = -\varepsilon, \quad \mathbf{k}'_\pm = \mathbf{k}_\mp, \quad \mathbf{s}'_\pm = \mathbf{s}_\mp, \quad s'_\pm = s_\mp.$$

† Under space inversion the spin conserves its helicity relative to the momentum.

Thus under time reversal as well as charge conjugation we have †

$$\varepsilon' = -\varepsilon, \quad s'_+ = s_-, \quad s'_- = s_+. \tag{11}$$

Let us first consider cases (c) and (d) when $s_+ = s_-$ and therefore

$$\lambda' = -\lambda = \varepsilon' s_{\varepsilon'} = \varepsilon' s'_{\varepsilon'}.$$

Hence it follows, after taking into account the transformation laws for the wave functions (9) and (10), that condition (5) conserves its invariant form under time reversal as well as under charge conjugation:

$$T = \text{const}, \qquad C = \text{const}. \tag{12}$$

These cases are equivalent to the new formulation proposed by Case [13]) (also see ref.[14])) in the Majorana theory of the neutrino if in the latter theory one assumes $k_0 = 0$.

Developing Pauli's work [12]), Case showed (see his eq. (8)) that the generalized Majorana equation can be obtained if one imposes on the Dirac wave functions the auxiliary condition

$$\psi_- = -i\alpha_2\rho_3\psi_+^{+\mathrm{T}}, \tag{13}$$

where $\psi^{+\mathrm{T}}$ is the transposed matrix and

$$\psi_\pm = \tfrac{1}{2}(1 \pm \rho_1)\psi. \tag{14}$$

It can easily be shown that conditions (13) and (14) can be derived from (1) if for any value of the energy ($\varepsilon = \pm 1$) we retain the sign before s and note that the wave function ψ also satisfies the Dirac equation

$$\varepsilon\rho_1\psi = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}}{i\sqrt{-\nabla^2}}\psi \tag{15}$$

and that the neutrino charge is zero:

$$\psi_{\varepsilon'=-1} = -i\alpha_2\rho_3\psi_{\varepsilon=1}^{+\mathrm{T}}.$$

From the physical standpoint the possibility of deducing some new results in the Majorana theory from conditions (c) or (d) is due to the fact that only one state is assigned to the neutrino ($\varepsilon = 1$) and one to the antineutrino ($\varepsilon = -1$), the spin directions of these states being identical ($s_+ = s_-$). Therefore, if the neutrino charge is zero these particles will be indistinguishable and the equation itself (as was just remarked) should be invariant under charge conjugation.

In cases (a) and (b), however, we have $s_+ = -s_-$ and hence

$$\lambda' = \lambda = \varepsilon' s_{\varepsilon'} = \varepsilon' s'_{\varepsilon'}.$$

It is easy to show with the help of the wave function transformation formulas (9) and (10), that the sign in the right hand side of (5) changes under time reversal ($T \neq \text{const}$) as well as under charge conjugation ($C \neq \text{const}$).

† Under time reversal the positon annihilation operator with $\varepsilon = -1$ becomes a negaton creation operator with $\varepsilon' = 1$; under charge conjugation it becomes the positon annihilation operator with $\varepsilon' = 1$.

This auxiliary condition will be invariant under combined inversion of the type

$$TC = \text{const}, \qquad I = \text{const}.$$

On the other hand, in cases (a) and (b) condition (1) conserves the value of s only for particles possessing identical energy signs since $s_+ = -s_-$. If the separation constant λ is to be the same for any of the values of $\varepsilon = \pm 1$, we should write instead of (1)

$$\left(\lambda - \frac{H}{|H|}\,\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|p|}\right)\psi = 0. \tag{16}$$

Hence for particles with a non-vanishing rest mass we obtain

$$\left(\lambda - \frac{k}{K}\rho_1 - \frac{k_0}{kK}\rho_3\boldsymbol{\sigma}\cdot\mathbf{k}\right)\psi = 0 \tag{17}$$

In particular for a neutrino ($k_0 = 0$) we obtain the condition

$$(\lambda - \rho_1)\psi = 0 \tag{18}$$

which yields either the results of Lee and Yang ($\lambda = 1$, case (a)) or the results of Gell-Mann and Feynman ($\lambda = -1$, case (b)). The experimental data indicate [15]) that in β^+-decay ($p \to n + e^+ + \nu$) the violation of symmetry is opposite to that observed in the case of β^--decay and hence there is much reason to believe that cases (a) and (b) are those that are actually realized, i.e. the neutrino (ν) and antineutrino ($\tilde{\nu}$) spins are oppositely directed in these cases. In order to give a final formulation of the theory and to verify the ICT-theorem it seems to be important to have data relating to asymmetry of antineutron decay and absorption of neutrinos or antineutrinos by neutrons.

In conclusion we may note that in our analysis parity non-conservation was related to the orientation of the neutrino spin characterized by the quantity s. Under space inversion the sign of s is to be changed, even though s in both cases refers to the same helicity. If one attempts to relate parity non-conservation in some way to the properties of space in which $I \neq \text{const}$, it will be necessary, in order to retain the Lüders-Pauli theorem, to introduce combined inversion of the type $IC = \text{const}$, $T = \text{const}$ (Landau [16])) although the possibility of $IT = \text{const}$, $C = \text{const}$ is also theoretically permissible.

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THE ELECTRICAL POLARIZABILITY OF THE MESON CLOUD IN THE NUCLEON

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An analysis of experiments on the scattering of fast electrons on hydrogen and deuterium, and slow neutrons on atoms leads to a result which is in contradiction to the conclusions of modern meson theory, namely, that the mean root square "electrical radius" of the neutron is practically zero. In this connection it becomes highly important to consider other effects which may reveal the "electromagnetic structure" of the nucleon. One such effect is the scattering of slow nucleons in an inhomogeneous electrical field (in particular, neutrons in the Coulomb field of the nucleus [1–3]), in which the cloud of charges of different signs in the nucleon becomes polarized and the nucleon is transformed into an electric dipole with an induced moment. The electrical polarizability of the charges in the nucleon manifests itself in the Compton effect and in the photoproduction of pions on nucleons [4]), as well as in the scattering of slow neutrons on atoms.

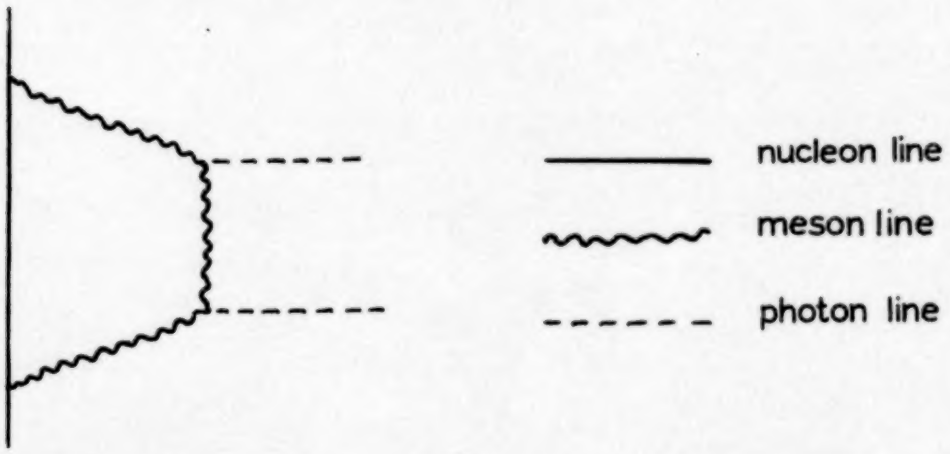

Fig. 1

We estimated the magnitude of the electrical polarizability of the meson cloud in the nucleon by means of the first approximation of Chew's theory [6]) (see fig. 1). The corresponding matrix element has the form:

$$M = -\left(\frac{f}{\mu}\right)^2 e^2\pi \int \frac{u(k)u(k')V(n-k)V(k'-n)}{\omega_k^2\omega_{k'}^2\omega_n^2} \cdot (\omega_k+\omega_n)(\omega_{k'}+\omega_n)k'_\mu k^\mu \frac{\mathrm{d}^3(kk'n)}{(2\pi)^9}, \quad (1)$$

with $\omega_k^2 = k^2+\mu^2$;

$$u(k) = \int u(x)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3x \quad (2)$$

denotes the form factor of the source, and

$$V(k) = \int V(x)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3x \quad (3)$$

the scalar electric potential. The remaining designations are the standard ones.

Since in the given case the vector part of the electromagnetic field is zero, the interaction with the electromagnetic field can be written down uniquely, and the theory is gauge invariant (cf. [7]).

For the case of a homogeneous field

$$V(k) = i(2\pi)^3 E_m \frac{\partial}{\partial k_m}\delta(\mathbf{k}) \quad (4)$$

the matrix element (1) may be represented in the form

$$M = -\tfrac{1}{2}\alpha E^2 \quad (5)$$

where the electrical polarizability

$$\alpha = \left(\frac{f}{\mu}\right)^2 e^2 \frac{2}{3\pi\mu}\int_0^\infty \frac{k^2}{\omega_k^8}\left\{u^2(k)(27k^4-34k^2\omega_k^2) - \left(\frac{\mathrm{d}u(k)}{\mathrm{d}k}\right)^2 4k^2\omega_k^4\right\}\mathrm{d}k \quad (6)$$

with $\omega_k^2 = k^2+1$ (we utilize the coordinate system for which $E_1 = E_2 = E_3$).

It may be expected, as in the computation of the magnetic moment of the nucleon and the potential of neutron-electron interaction [9]), that the higher terms in the series development in the constant f/μ do not essentially affect the result (6). One finds

$$\text{for} \quad u(k) = \frac{1}{1+(k/5.6)^2}: \qquad \alpha = 1.6\times 10^{-42}\,\mathrm{cm}^3$$

$$\text{for} \quad u(k) = \exp\left[-\frac{k^2}{2(5.6)^2}\right]: \quad \alpha = 1.8\times 10^{-42}\,\mathrm{cm}^3.$$

(The same form functions have been chosen as in ref. [8]).) These values of α were obtained for $f^2/\hbar c = 0.08$. The results of the computations are but slightly sensitive to the given choice of $u(k)$.

The computed value of α is close to that obtained by Baldin from an analysis of experiments on the photoproduction of pions and the Compton effect on the nucleon [4])

$$4\times 10^{-43}\,\mathrm{cm}^3 \leqq \alpha \leqq 1.4\times 10^{-42}\,\mathrm{cm}^3$$

and is considerably less than the value of α obtained by Alexandrov from experiments on the scattering of slow neutrons [2-3]) ($\alpha \approx 5\times 10^{-41}\,\mathrm{cm}^3$). A more thorough analysis of these experiments is necessary. We observe that the effects involving the polarizability α rapidly increase with a decrease in the energy of the scattered neutron.

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NON-LOCAL STRUCTURE OF QUANTIZED FIELD THEORIES OF THE SECOND KIND

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Abstract: The transcendental vacuum expectation values arising from quantized field theories of the second kind are shown to impart to those theories a non-local structure which is characterized by an indeterminacy in space-time of the light-cone, indefinite metric in Hilbert space and non-invariance under similarity transformations in space-time.

1. Introduction

Relativistic quantum field theories can be formulated in terms of vacuum expectation values of products of field operators $F_n(x_1, \ldots, x_n) = \langle \phi(x_1) \ldots \phi(x_n) \rangle_0$: Given a set of Lorentz invariant distributions $F_n (n = 1, 2, \ldots)$ satisfying certain basic axioms, the field theory pertaining to this set is determined almost completely [1]). In the present paper we discuss some properties of vacuum expectation values determined by quantized field theories of the second kind. These theories (which include derivative and Fermi couplings) seem to require a deviation from the conventional axiomatic scheme of relativistic quantum theory on account of the non-tempered behaviour of the associated spectral functions. The latter is due to the fact that, for example, in derivative coupling theories which automatically give strong coupling at high energies, diagrams whose vertices are accumulation points of boson lines play the decisive part in the S-matrix [2]). The corresponding vacuum expectation values are transcendental functions of Δ_F which possess essential singularities at the light-cone. Their Fourier transforms and consequently the spectral functions are non-tempered distributions for large momenta. We shall show that the transcendental vacuum expectation values impart to theories of the 2nd kind certain non-local aspects [3]).

The transcendental nature of the propagators determined by field theories of the 2nd kind was first recognized by Cooper [4]) for the case of simple derivative coupling theories which possess *exact explicit* solutions

† On leave of absence from the Technische Hochschule, Aachen, Germany. Work performed under contract with the Brasilian National Research Council.

(propagators such as $\exp[\lambda\delta_-(t)]$). Chisholm [5]) † has shown that the S-matrix of the neutral ps—pv theory can be formulated in terms of transcendental propagators such as $\exp[g_0^2\Delta_F(x_\nu^2;\mu^2)]$ (cf. eq. (12) below). Although no closed form solutions are known in this case, it seems likely that the essential transcendental structure of the vacuum expectation values will be preserved also in exact solutions, if those exist. Glaser's explicit solution of the Fermi-interaction problem in two-dimensional space-time [12]) †† supports the conjecture that also relativistic Fermi couplings give rise to transcendental propagators and thus to non-local effects [7]) †††. The importance of derivative couplings in the weak boson-fermion interactions has recently been stressed by Umezawa *et al.*[11]).

2. Multi-Meson Propagators *

2.1. THE FUNDAMENTAL LENGTH

Transcendental vacuum expectation values, i.e., transcendental functions of Δ_F, Δ^+ etc., are defined by formal series in the vector space K of distribution products introduced by König [8]). It should be pointed out, however, that series expansions in terms of multiple propagators $[g_0^2\Delta_F(x_\nu^2;\mu^2)]^n$ cannot be considered as coupling constant expansions even if they arise from formal perturbation analysis. This is an immediate consequence of the fact that the "local part" of $[\Delta_F]^{n+1}$ contains the relativistic multipole $\delta^{(n)}(x_\nu^2)$ according to **

$$\{\mathrm{Im}[\Delta_F(x_\nu^2;\mu^2)]^{n+1}\}_{\mu=0} = \mathrm{Im}[D_F(x_\nu^2)]^{n+1} = a_n\delta^{(n)}(x_\nu^2)+b_n\,\Box^{n-1}\delta^4(x), \quad (1)$$

a_n, b_n being numerical constants (cf. eq. (10)). The Fourier transform of $\delta^{(n)}(x_\nu^2)$ $(n \geqq 1)$ is given by [9])

$$\mathbf{F}\delta^{(n)}(x_\nu^2) = -(k_\nu^2)^{n-1}[\log|\gamma k_\nu^2/4|-\psi(n)]/2^{2n+2}\pi^3(n-1)! \quad (2)$$

† See also ref. [6]). The procedures of the last mentioned authors are not free from objections, however. The statement that the delta-function in Δ_F does not contribute to $\exp(\Delta_F)$ is incorrect.

†† Thanks are due to Dr. Glaser for communications in advance of publication.

††† Suppose the exponential dependence on the coupling constant $\sqrt{g_F}$ (dimension of length) of the solution of the relativistic Fermi interaction to be the same as that of Glaser's solution (with dimensionless coupling constant g_G). Then one arrives at a Z-factor of the form $Z^{\frac{1}{2}} = \exp(ig_F I)$ with $I = \infty$. Hence, for dimensional reasons, $I \sim (1/x_\nu^2)_{x_\nu=0}$ and Z would have the same form as in Cooper's models and in ps—pv theory. This hints at propagators with essential singularities of the type considered below (cf. eq. (12) where now $g = \sqrt{g_F} \approx 10^{-17}$ cm) [11]).

* The product problem discussed below can be circumvented by using spectral-representation methods. In particular, the right side of eq. (10) is essentially equivalent to $\int_0^\infty \mathrm{d}m^2\,\rho(m^2)\Delta_F(\xi_\nu^2;m^2)$ with the spectral function $\rho(m^2) = (m^2)^{n-2}$. Cf. W. Güttinger and C. Braga, to be published.

** Eq. (1) may be proved by asymptotic expansions, independently of the particular structure of the vector space K. For example, $D_F(x_\nu^2)\cdot D_F(x_\nu^2-\varepsilon) = (1/2\pi^2)\mathrm{d}D_F/\mathrm{d}(x_\nu^2)+\mathrm{o}(\varepsilon)$, D_F being given after eq. (11). We use $x_\nu^2 = x_0^2-\mathbf{x}^2$.

where $\mathbf{F}T(x) = (1/2\pi)^4 \int \mathrm{d}^4x \exp(-ik_\nu x_\nu) T(x)$, $\psi(n) = \Gamma'(n)/\Gamma(n)$, and $\log \gamma = C$ is Euler's constant. On account of the logarithmic term in eq. (2), the variables x_ν and k_ν cannot be identified with physical space-time and momentum coordinates except through the explicit introduction of a fundamental length, say, g, such that eq. (2) is replaced (via a change of variables) by the dimensionally correct relation

$$\mathbf{F}\delta^{(n)}(\xi_\nu^2) = -(p_\nu^2)^{n-1}[\log|\gamma p_\nu^2/4| - \psi(n)]/2^{2n+2}\pi^3(n-1)! \qquad (3)$$

where now $\mathbf{F}T(\xi) = (1/2\pi)^4 \int \mathrm{d}^4\xi \exp(-ip_\nu \xi_\nu) T(\xi)$ and

$$\xi_\nu = (x/g)_\nu, \quad p_\nu = (kg)_\nu \qquad (4)$$

are dimensionless variables. Since the introduction of the length g must be due to the interaction, the coordinates x_ν possess physical length dimension only through the interaction concept. Separation of x_ν and k_ν from g is of course inadmissible.

Hence, only multiple propagators of the type $[D_\mathrm{F}(\xi_\nu^2)]^n$, $[\Delta_\mathrm{F}(\xi_\nu^2; (g\mu)^2)]^n$ are meaningful and it is impossible to take g^2 outside the products $[g^2 D_\mathrm{F}]^n$, $[g^2\Delta_\mathrm{F}]^n$ (except that $g = 1/\mu$ in the latter product). The transcendental propagators therefore are to be regarded as series expansions with respect to a set of "basis vectors" $[\Delta_\mathrm{F}(\xi_\nu^2)]^n \in \mathrm{K}$, the series being no coupling constant expansions whatever. Further, we see that in field theories of the 2nd kind only vacuum expectation values of the type $\langle T([g\phi(x)]^n [g\phi(x')]^n)\rangle_0$ and interactions such as $L_{\mathrm{ps,pv}} = \bar{\psi} m[\exp(2\gamma_5[g\phi]) - 1]\psi$ are physically meaningful, in which $[g\phi(x)] = \Phi(\xi)$ forms one single unit that cannot be split into a product of two independent quantities g and $\phi(x)$ (coupling constant and meson field).

The logarithmic term in eq. (3) appears as a factor in the Fourier transform of any infinite multipole series and, therefore, in that of any transcendental propagator. Hence, the length g in (3) (in particular, the length $g = g_0 = \sqrt{(4\pi)}f/\mu$ (coupling constant) in ps-pv theory) determines a "critical distance", $|x_\nu^2| \approx |g^2|$, $|k_\nu^2| \approx |g^{-2}|$, in the theories under consideration. This applies to both the infinite series and the individual terms (multipoles and multiple propagators), as may also be inferred, for example, from the functional representation of $\delta'(\xi_\nu^2)$:

$$\delta'(\xi_\nu^2)[\varphi(\xi)] = \lim_{\varepsilon \to 0} \int \mathrm{d}^4\xi \{\delta'(\xi_\nu^2 - \varepsilon^2) + \tfrac{1}{2}\pi \log \varepsilon^2 \delta(\xi)[\delta(\xi_0 - \varepsilon) + \delta(\xi_0 + \varepsilon)]\}\varphi(\xi) \qquad (5)$$

where $\varphi(\xi)$ is an indefinitely differentiable testing function.

From eq. (3) (or, for $n = 1$, directly from eq. (5)) we see that $\delta^{(n)}(\xi_\nu^2)$ $(n \geqq 1)$ is not homogeneous with respect to similarity transformations in space-time (change of scale $x_\nu \to x_\nu/\alpha$ or $\xi_\nu \to \xi_\nu/\alpha$) since for any real number $\alpha \neq 0$ and any integer $n \geqq 0$,

$$\delta^{(n)}(\xi_\nu^2/\alpha^2) = (\alpha^2)^{n+1}\delta^{(n)}(\xi_\nu^2) - \frac{(-\alpha^2)^{n+1}\pi \log \alpha^2}{2^{2n-2}(n-1)!}\Box^{n-1}\delta^4(\xi). \qquad (6)$$

Hence, also the infinite multipole series $H(\xi_\nu^2) = \sum_{n=0}^{\infty} c_n \delta^{(n)}(\xi_\nu^2)$, are not invariant under similarity transformations in space-time continuum R_4. The domain of non-invariance of H generally consists of a finite non-point-like region in R_4 while that of the multipoles is confined to the origin of the light-cone.

2.2. STRUCTURE OF VECTOR SPACE

König's results [8]) imply that the vector space K of products of distributions is essentially isomorphic to the vector space D′ of distributions for a given, fixed coordinate system. This implies that the values in K of Lorentz invariant multiple propagators depend on the particular coordinate unit employed in Minkowski space $R_4(\xi)$ and that invariance of the propagators under similarity transformations in R_4 cannot longer be ensured. This is already a property of the multipoles involved in the propagators, however.

The structure of K may now be determined by the following two postulates:

(I) A Fourier operator $\mathscr{F}$ defines an automorphism in K such that (i) $\mathscr{F}T_i = \mathbf{F}T_i$ if $T_i(\xi_\nu^2)$ is a Lorentz invariant distribution in K and that (ii) the relation

$$\Box_p \mathscr{F}T = -\mathscr{F}(\xi_\nu^2 \circ T) \qquad (7)$$

be valid for each element

$$T = T_1 \circ T_2 \circ \ldots \circ T_n = T_1 \circ (T_2 \circ (\ldots \circ (T_{n-1} \circ T_n) \ldots)$$

of K, the symbol $\circ$ denoting the product in K.

(II) If $f(\xi_\nu^2)$ is an indefinitely differentiable Lorentz invariant function, then in K

$$f \circ (T_1 \circ T_2 \circ \ldots \circ T_n) = T_1 \circ T_2 \circ \ldots \circ T_{n-1} \circ (f \cdot T_n) \qquad (8)$$

where the multiplicative dot in $f \cdot T_n$ denotes the ordinary product.

By choosing $f(\xi_\nu^2) = \xi_\nu^2$ and using the relation

$$\xi_\nu^2 \cdot \delta^{(n)}(\xi_\nu^2) = -n\delta^{(n-1)}(\xi_\nu^2) + \frac{(-1)^n \pi}{2^{2n-4}(n-2)!}\Box^{n-2}\delta^4(\xi), \qquad (9)$$

we obtain in K (with $T_i = D_F$)

$$[D_F(\xi_\nu^2)]^n = (1/2\pi^2)^{n-1} D_F^{(n-1)}(\xi_\nu^2)/(n-1)! + iS_n(\xi) \qquad (10)$$

with

$$S_n(\xi) = \left(-\frac{1}{2\pi^2}\right)^{n-1} \cdot \frac{[\psi(n) + \log \gamma - 1]}{2^{2n-3}(n-1)!(n-2)!}\Box^{n-2}\delta^4(\xi). \qquad (11)$$

Here, $D_F = D_F(\xi_\nu^2) = -[\mathcal{P} \cdot v \cdot (1/\xi_\nu^2) + i\pi\delta(\xi_\nu^2)]/2\pi^2$ and $D_F^{(n)}(\xi_\nu^2) = \mathrm{d}^n D_F/\mathrm{d}(\xi_\nu^2)^n$. By considering D_F as a boundary value of an analytic function, $D_F = (-1/2\pi^2)\lim_{\epsilon \to 0}(\xi_\nu^2 - i\varepsilon)^{-1}$, the first term on the righthand of eq. (10)

follows by taking the individual limits in the product uniformly. The term S_n in (10) is essentially a consequence of eq. (9) (which follows directly from eq. (3)).

3. Transcendental Propagators

Both the Hamiltonian approach and the integral equations of Lehmann *et al.*[10]) lead to a formulation of field theories of the 2nd kind in terms of transcendental propagators which, as has been shown above, are to be defined by series expansions in K with respect to infinite sets of multiple operators $[\Delta_F(\xi_\nu^2)]^n$, $[\Delta^+(\xi_\nu^2)]^n$ etc. †. By virtue of eq. (1), any series $G(\Delta_F) = \sum_{n=0}^{\infty} a_n[\Delta_F(\xi_\nu^2)]^n$ contains an infinite series of multipoles $H(\xi_\nu^2) = \sum_{n=0}^{\infty} c_n \delta^{(n)}(\xi_\nu^2)$ which represents a non-local quantity that determines the global structure of the propagator in the same sense as $\delta(x_\nu^2)$ in Δ_F characterizes the local structure of the free particle function. $H(\xi_\nu^2)$ generally is not localized at the light-cone $\xi_\nu^2 = 0$, the type of non-localizability being determined by the order of growth ρ (in $|p_\nu^2|^{\frac{1}{2}}$) of the Fourier transform $\mathfrak{F}H$. Since $G(\Delta_F)$ and $G(D_F)$ contain essentially the same multipole series H, the non-local structure of the vacuum expectation values $G(\Delta_F)$ and $G(\Delta^+)$ does not sensitively depend on the mass parameter μ involved in Δ_F, Δ^+.

As an example typical of derivative coupling theories we consider the propagator

$$G_0(D_F) = M^2[\exp(2\gamma_5^1\gamma_5^2 D_F(\xi_\nu^2)) - 1] \tag{12}$$

($\xi = \xi_1 - \xi_2$), the analogue of the Møller interaction in the zero meson mass limit of neutral ps-pv theory [3,5]). $G_0(D_F)$ might also be viewed as the relativistic analogue of Cooper's closed form propagator exp $(\lambda\delta_-(l))$ [4]). $G_0(D_F)$ is defined by its series expansion, $G_0 = I_0 + iH_0$, the associated multipole series H_0 being given (according to (10)) by

$$H_0 = -\pi M^2 \sum_{n=0}^{\infty} \frac{(\gamma_5^1\gamma_5^2/\pi^2)^{n+1}}{n!(n+1)!}\,\delta^{(n)}(\xi_\nu^2) + R(\xi) \tag{13}$$

where

$$R(\xi) = M^2 \sum_{n=1}^{\infty} (2\gamma_5^1\gamma_5^2)^n S_n(\xi)/n!. \tag{14}$$

The length g in $\xi_\nu = (x/g)_\nu$ now coincides with the coupling constant $g_0 = \sqrt{(4\pi)}f/\mu$ of the linear representation of the neutral ps-pv interaction Lagrangian $L' = ig_0\bar{\psi}\gamma_5\gamma_\mu\partial_\mu\phi\psi$. However, g no longer plays the role of a coupling constant in the customary sense.

We disregard for the moment the contribution of R to H_0 and also neglect the 4-dimensional structure of the multipoles. That is, we treat the multipoles as if they were one-dimensional delta-functions, and then arrive at the functional representation

† This holds also for closed form propagators.

$$H_0(\varphi) = \pi M^2 \sum_{n=0}^{\infty} \frac{(-\gamma_5{}^1\gamma_5{}^2/\pi^2)^{n+1}}{n!(n+1)!} \varphi^{(n)}(0) \tag{15}$$

with the testing function $\varphi = \varphi(\xi_\nu{}^2)$, $\delta^{(n)}(\xi_\nu{}^2)[\varphi(\xi_\nu{}^2)] = (-1)^n \varphi^{(n)}(0)$. Assuming φ to be analytic in the complex z^2-plane, $z^2 = (\xi_\nu{}^2 + i\eta)$, we obtain

$$H_0(\varphi) = \frac{1}{2\pi i} \int_C \mathrm{d}(z^2) h_0(z^2) \varphi(z^2) \tag{16}$$

where

$$h_0(z^2) = \pi M^2 [\exp(-\gamma_5{}^1\gamma_5{}^2/\pi^2 z^2) - 1], \tag{17}$$

C being any circle round the origin of the z^2-plane.

Inclusion of the term $R(\xi)$ in (13) produces a branch cut and the closed form representation of H_0 is no longer trivial. If one takes into account the true 4-dimensional structure of the multipoles in (13) one sees immediately that, by virtue of eq. (6), H_0 is not invariant with respect to similarity transformations in space-time. This important property of H_0 is of course lost in the simplified representations (15) and (16). We remark incidentally that the "potentials" derived from H_0 by time-integration reveal a repulsive core.

H_0 obviously is *not* localized at the light-cone $\xi_\nu{}^2 = 0$ since the value of $H_0(\varphi)$ depends on, and is likewise determined by, the values of $\varphi(\xi_\nu{}^2)$ on any given infinite and bounded set of hyperboloids $(\xi_\nu{}^2)_i \neq 0$ round $\xi_\nu{}^2 = 0$. However, no definite circle C can be fixed outside of which H_0 vanishes identically and, therefore, we may define the support (carrier) of H_0 to be that domain outside of which $H_0(\varphi)$ does not sensitively depend on the values of φ. This domain consists of the volume $|\xi_\nu{}^2| \lesssim 1$, i.e., of the space-time region determined by $|x_\nu{}^2| \lesssim |g^2|$. The meaning of g as a critical length may also be inferred from the Fourier transform of H_0 which behaves asymptotically for large $|p_\nu{}^2| = |(kg)_\nu{}^2|$ as

$$\mathfrak{F}H_0 \sim \log |p_\nu{}^2| \cdot \exp(\beta |p_\nu{}^2|)^{\frac{1}{3}}, \tag{18}$$

β being a numerical *complex* constant with $|\beta| \approx 1$. $\mathfrak{F}H_0$ has order of growth $\rho = \frac{2}{3}$.

4. Discussion

The Fourier transforms of $G_0(D_F)$, $G_0(D^+)$ and H_0 begin to oscillate at $|p_\nu{}^2| \approx 1$, i.e., at $|k_\nu{}^2| \approx |g^{-2}|$, and oscillate with increasing frequency (amplitude depending on arg $(p_\nu{}^2)$) as $|p_\nu{}^2| = |(kg)_\nu{}^2| \to \infty$, but are slowly increasing or decreasing in the region $|p_\nu{}^2| = |(kg)_\nu{}^2| \lesssim 1$. In consequence of the non-tempered behaviour of the momentum space operators also the associated spectral functions are non-tempered. This is a general property of vacuum expectation values whose Fourier transforms have order of growth

$\rho > 0$. The oscillations, for $|k_\nu^2| \gtrsim |g^{-2}|$, of the vacuum expectation values in momentum space imply that Wightman's positive-definiteness conditions in Hilbert space [1]) must fail to hold, since the suppositions of the generalized Bochner theorem (slowly increasing, positive Fourier transforms) are no longer satisfied. Since, however, these oscillations are restricted to the region $|p_\nu^2| \gtrsim 1$, positive-definiteness conditions can be established in the domain $|(kg)_\nu^2| \lesssim 1$, i.e., for $|x_\nu^2| \gtrsim |g^2|$, by means of a space of testing functions which vanish for $|x_\nu^2| \lesssim |g^2|$. Indefinite metric and accausal effects are therefore confined to small space-time regions. The weaker the interaction (i.e., the smaller g), the smaller the size of those regions, the limit $g \to 0$ being highly non-uniform, however. The domain of non-invariance of H_0 under similarity transformations would seem to coincide with the support of H_0.

The support of H_0 and, more generally, the support of the multipole series associated with any transcendental vacuum expectation value with $0 < \rho < 1$, is non-localizable in the proper sense of the word [3]): The non-localizability of the support of $H (0 < \rho < 1)$ may be viewed as an indeterminacy in space-time $R_4(\xi)$ of the light-cone. With regard to $G_0(D_F)$, $G_0(\Delta_F)$, this indeterminacy means something like a statistical spread of the positions of two interacting point-like fermions over a space-time volume of size $|x_\nu^2| \lesssim |g^2|$ (e.g., of size $|x_\nu^2| \lesssim \mu^{-2}$ if $f^2 \approx 0.08$ in $g = g_0 = \sqrt{(4\pi)f/\mu}$). The theory still is formally Lorentz invariant since each finite partial series of H_0 is so. Obviously no structure function is being encountered with.

Order of growth $0 < \rho < 1$ has so far been ensured for derivative coupling theories. Vacuum expectation values with order of growth $\rho = 1$ possess H-series the support of which is strictly localizable at some hyperboloid $z_0^2 = (\xi_\nu^2 + i\eta)_0 \neq 0$ (dislocation of the support). Such a situation is encountered with in the unphysical region of dispersion relation theory. It is not known whether Fermi interactions belong to one of the classes considered (cf., however, the footnote ††† on p. 430 and ref.[11])).

The non-invariance of H under similarity transformations in R_4 would seem to hint at non-homogeneity of space-time and thus might be related to deviations from flat Minkowski space if one adopts the point of view that physical space-time must deduce its properties from the interaction concept. This non-invariance likewise points to a failure of the concept of a renormalization group.

Summing up, it may be stated that transcendental vacuum expectation values and non-tempered spectral functions determine quantized field theories which are characterized by non-localizability, indefinite metric and non-invariance with respect to similarity transformations in space-time.

Further details will be published elsewhere.

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NUCLEAR CROSS SECTIONS FOR THE SCATTERING OF NEUTRONS AND PROTONS

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Abstract: Cross sections and polarization effects are calculated for the scattering of neutrons and protons by Al, Fe and Pb. The calculations are based on an optical model with rounded profile and spin orbit coupling, and are carried out in WKB approximation above 300 MeV and by exact computation of the phase shifts at lower energies. The nuclear radii determined from the high energy data are smaller than those reported in earlier studies of nuclear cross sections, and are close to the radii determined by the analysis of electron scatterings at high energies. The radius obtained from the 300 MeV analysis also yields good results for the scattering of neutrons by Zr at 7 MeV. At all energies polarization data are found to be essential for the determination of the parameters of the model.

1. Introduction

In the present note we report the results of calculations on the angular distribution and polarization of nucleons elastically scattered by Al, Fe and Pb at ≈ 300 MeV, and by Zr at 7 MeV. The calculations are based on an optical model with rounded profile and spin-orbit coupling, and are carried out in WKB approximation for energies above 300 MeV and by exact computation of the phase shifts at lower energies.

The model provides a good description of proton scattering and polarization at 300 MeV. The comparison with observation [1,2]) determines the nuclear radius within narrow limits, and, moreover, the radius deduced from the 300 MeV analysis also yields good results for the description of scattering at lower energies, in the neighbourhood of 7 MeV.

We find that the nuclear radii determined from the high energy data are smaller than those recorded previously in studies of nuclear cross sections [3]), and are close to the radii determined by the analysis of electron scattering at high energies [4]).

At all energies polarization data are essential for the determination of the parameters of the model. It is in fact found that the cross section data alone can be described by a wide range of nuclear radii and optical parameters. Even with polarization data, the number of parameters in the model is sufficiently great so that the parameter assignments have an appreciable latitude, as seen by the probable errors listed in tables 1 and 2.

TABLE 1

Profile parameters at 300 MeV. Lengths are given in units of 10^{-13} cm

	Al^{27}	Fe^{56}	Pb^{208}
a	0.65 ± 0.10		
$r_0/A^{\frac{1}{3}}$	1.01 ± 0.04	1.12 ± 0.04	1.20 ± 0.10
Electron Scattering [a])			
a	0.54		
$r_0/A^{\frac{1}{3}}$	Mg^{24} : 0.99 Si^{28} : 0.97	V^{51} : 1.07 Co^{59} : 1.05	1.09

[a]) See ref.[4]).

TABLE 2

Nuclear potential strengths at 300 MeV. The potentials are given in MeV, and wave numbers (k_1, K) in units of 10^{13} cm^{-1}. HJ refers to the results of the present analysis

	Central				Spin-Orbit	
	Real		Imaginary		Real	Imaginary
	V_1	k_1	V_2	K	U_1	U_2
HJ	14 ± 8	0.09 ± 0.05	29 ± 5	0.35 ± 5	0.45 ± 0.15	0.15 ± 0.15
Bethe[9])	5.7	—	22	—	0.55	−0.17

2. Description of the Model

We represent the density distribution of nuclear matter by the Saxon-Woods profile [5]),

$$\rho = \frac{\rho_0}{1+\exp\left(\frac{r-r_0}{a}\right)} \tag{1}$$

which describes a uniform density in the region $r \lesssim r_0-a$ and a gradual decrease at the surface between r_0-a and r_0+a. The thickness of the diffuse boundary is thus $2a$. The constant a is assumed independent of atomic number A. For r_0 we expect an $A^{\frac{1}{3}}$ variation:

$$r_0 \approx R_0 A^{\frac{1}{3}}.$$

The extent of the diffuse region is presumably given by the range of nuclear forces, or approximately 1×10^{-13} cm. Thus, we expect $a \approx 0.5$

$\times 10^{-13}$ cm. Earlier examinations of total neutron cross sections indicate that the appropriate values of R_0 lie in the neighbourhood of 1.4×10^{-13} cm [6]).

The nuclear potential is represented by a complex central well whose form follows the nuclear density distribution, plus a complex spin-orbit interaction:

$$V = (V_1+iV_2)\rho(r)+4ar_0(U_1+iU_2)\frac{1}{r}\frac{\mathrm{d}\rho}{\mathrm{d}r}\boldsymbol{\sigma}\cdot\mathbf{l} \tag{2}$$

where $\rho(r)$ is defined by (1). The constants in the second term of (2) are chosen so that U_1+iU_2 is the maximum value of the spin-orbit coupling on the nuclear surface.

We relate V_1 and V_2 to a complex propagation constant $k_1+\frac{1}{2}iK$ in the nuclear interior by the formula

$$V_1+iV_2 = (k+k_1+\tfrac{1}{2}iK)^2-k^2 \tag{3}$$

where k is the de Broglie wave number of the incident nucleon. Appropriate ranges of values for the nuclear opacity K and the index of refraction of nuclear matter $1+(k_1/K)$ are suggested by their approximate relations to the nucleon-nucleon scattering values:

$$K = \rho_0\sigma_{\text{N-N}}, \quad k_1 = \frac{\pi}{k}\rho_0 f_{\text{N-N}}(0). \tag{4}$$

Here ρ_0 is the particle density at the centre of the nucleus, $\sigma_{\text{N-N}}$ the total N-N (nucleon-nucleon) cross section, and $f_{\text{N-N}}(0)$ the forward N-N scattering amplitude. For reasonable values of ρ_0 and $\sigma_{\text{N-N}}$ at 300 MeV, we find $K \approx 0.28\times 10^{13}$ cm^{-1}. The forward amplitude is not determined unambiguously by N-N data, but may be computed from a potential model for the N-N interaction. Using the hard core potential, we find [7]) $k_1 \approx 0.10\times 10^{13}$ cm^{-1} at 315 MeV. These results serve as suitable starting values for the subsequent variation of optical parameters.

3. Determination of the Parameters

The data available for comparison with our calculations consist of the total cross section σ, inelastic or absorption cross section σ_a, angular distribution $\mathrm{d}\sigma/\mathrm{d}\Omega$, and polarization $P(\theta)$. Certain characteristics of the scattering are particularly sensitive to variations in a single parameter of the model, and this sensitivity enables us to make approximate parameter assignments in sequence, according to the following procedure: (a) the measured width of the diffraction pattern in $\sigma(\theta)$ fixes r_0 or r_0+a; (b) K is adjusted for agreement with the measured values of the absorption cross section; (c) the spin-orbit strengths, U_1 and U_2 are determined by the form and magnitude

of the polarization; (d) a is varied (always with appropriate readjustments in r_0 as required by (a) and (b)) to bring the large-angle scattering into agreement with the data, in particular with respect to the height of the secondary diffraction maximum.

No single aspect of the scattering is sufficiently sensitive to k_1 to permit its separate determination, and the value of this parameter must be readjusted at each of the preceding steps.

4. Results and Discussion

4.1. Al

The parameters determined in this way for optimum agreement with the data on Al are given in tables 1 and 2. The calculated cross section and polarization are compared with observation in figs. 1 and 2, respectively,

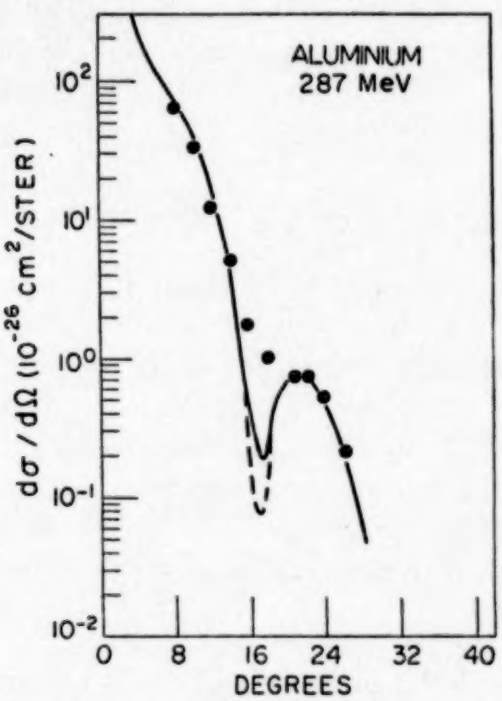

Fig. 1. Scattering of protons by Al at 287 MeV. The circles indicate data taken from ref.[2]). The calculated distribution is based on the parameters $r_0 = 1.05$, $a = 0.65$, $V_1 = 20$ MeV, $V_2 = 25$ MeV, $U_1 = 0.5$ MeV, $U_2 = 0.15$ MeV. The experimental angular resolution of 1° has been folded into the calculations. The dashed line indicates the original curve.

for typical parameter assignments. The agreement with the data is relatively good. The discrepancies at the first diffraction minimum may be attributed to an inelastic contribution, and perhaps to an experimental underestimate of angular resolution.

4.2. Fe AND Pb

The optical constants and surface thickness should be independent of nuclear radius. The cross sections and polarizations for Fe and Pb were therefore calculated using the values of these parameters deduced from the

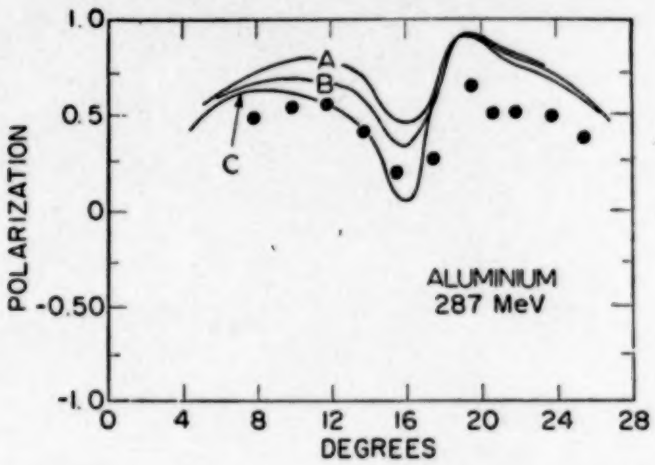

Fig. 2. Polarization of protons by Al at 287 MeV. The circles indicate data taken from ref.[3]). The three curves illustrate the effect of variations in U_2, and correspond to the following values of this parameter: (A) 0; (B) 0.15 MeV (C) 0.30 MeV. The remaining parameters are as in fig. 1. The experimental angular resolution of 1° has been folded into the calculations.

Al data, and the results are compared with observation in figs. 3—5. It is seen that the agreement in the case of Fe is as good as that for Al, the only significant discrepancies being those already noted in the lighter element. However, the model provides a poor description of the scattering by Pb, for which the calculations exceed the measured values by a factor 5 at the angle of the first of the secondary maximum. In an effort to remove the discrepancy the calculations were repeated for Pb with an extensive variation of optical and profile parameters, but no substantial improvement could be effected while maintaining the observed values of σ and σ_a. This disagreement represents a serious defect in the present model.

For both Fe and Pb the radius was adjusted to agree with the width of the angular distribution and with the measured total cross section. The corresponding values of r_0 are given in table 1.

4.3. Zr

Exact phase shift calculations have been carried out for neutrons on Zr at 7 MeV. Curve I of fig. 6 shows the comparison with experiment [8]) for the parameters $V_1 = 56$ MeV, $V_2 = 4.5$ MeV, $U_1 = 0.4$ MeV, $U_2 = 0$, $r_0 = 1.11A^{\frac{1}{3}}$, and $a = 0.80$. The profile parameters are close to those determined from the preceding analysis of high energy data. Similar calculations have been carried out by Beyster *et al.*[8]) for a model without spin-orbit

coupling. Their results are shown in curve II of fig. 6. It is seen that the addition of a spin-orbit coupling partially eliminates the diffraction minima, an effect which is usually attributed to compound elastic scattering at low

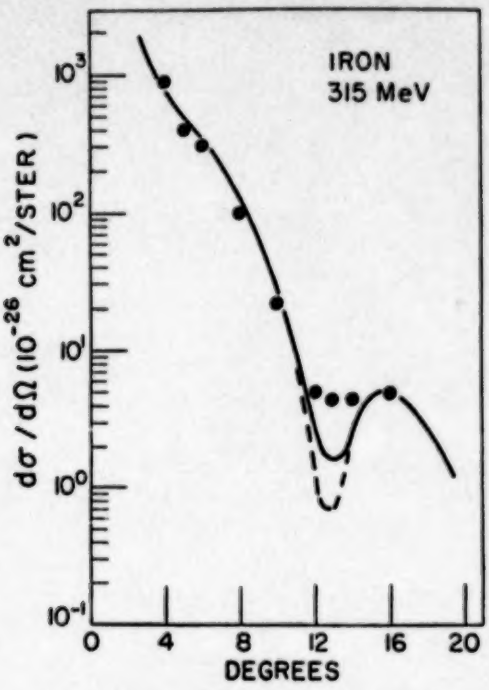

Fig. 3. Scattering of protons by Fe at 315 MeV. The circles indicate data taken from ref.[2]). The calculated distribution is based on the parameters $r_0 = 1.09A^{\frac{1}{3}} \times 10^{-13}$ cm, $a = 0.65 \times 10^{-13}$ cm, $V_1 = 20$ MeV, $V_2 = 24$ MeV, $U_1 = 0.3$ MeV, $U_2 = 0.1$ MeV. The experimental angular resolution of 1° has been folded into the solid curve. The dashed line indicates the original calculations.

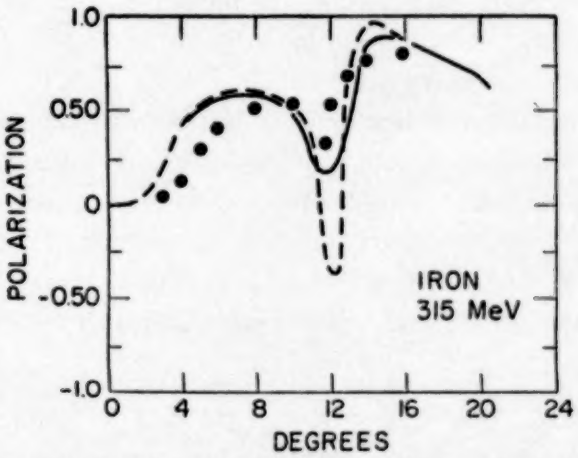

Fig. 4. Polarization of protons by Fe. The circles indicate data taken from ref.[2]). All parameters are as in fig. 3. The experimental angular resolution of 1° has been folded into the solid curve. The dashed line indicates the original calculations.

energies. The compound elastic and spin-orbit contributions cannot be separated by analysis of the angular distributions alone, but polarization data should suffice to settle this interesting question.

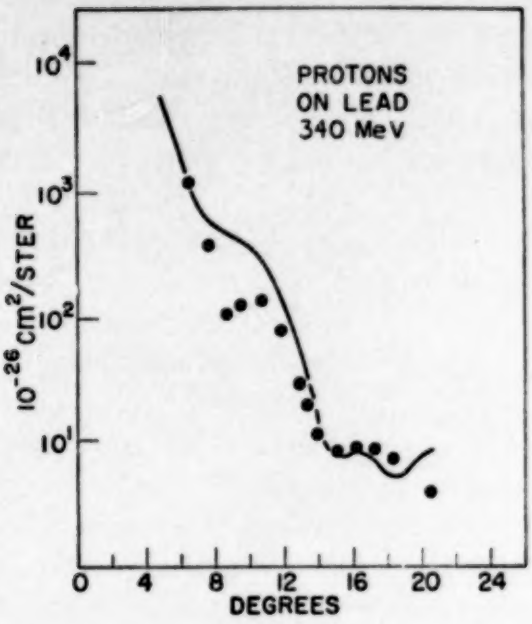

Fig. 5. Scattering of protons by Pb at 340 MeV. The circles indicate data taken from ref.[1]). The calculated distribution is based on the parameters $V_1 = 20$ MeV, $V_2 = 25$ MeV, $U_1 = 0.2$ MeV, $U_2 = 0$.

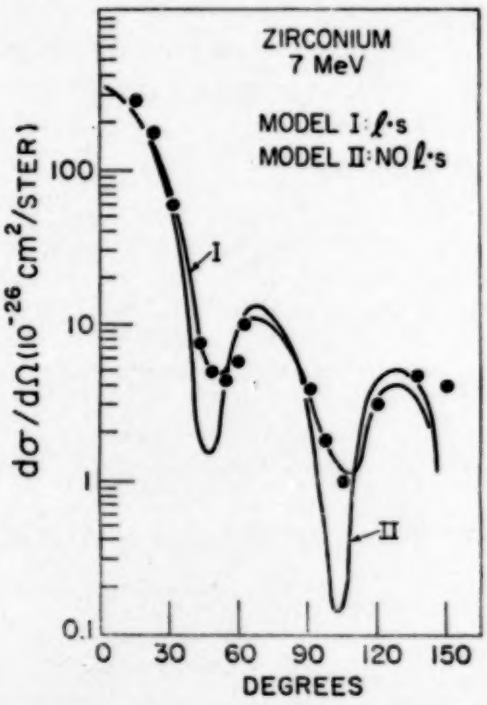

Fig. 6. Scattering of neutrons by Zr at 7 MeV. The circles indicate data taken from ref.[8]). Curves I and II represent models with and without spin-orbit couplings, respectively. The parameters for each model are: (I) $r_0 = 1.11A^{\frac{1}{3}} \times 10^{-13}$ cm, $a = 0.80 \times 10^{-13}$ cm, $V_1 = 56$ MeV, $V_2 = 4.5$ MeV, $U_1 = 0.4$ MeV, $U_2 = 0$; (II) $r_0 = 1.33A^{\frac{1}{3}} \times 10^{-13}$ cm, $a = 0.5 \times 10^{-13}$ cm, $V_1 = 45$ MeV, $V_2 = 4.5$ MeV, $U = 0$.

4.4. COMPARISON WITH CHARGE DISTRIBUTION PROFILES

In table 1 we make a comparison between nuclear radii determined by the present analysis and the charge distribution radii deduced from the analysis of electron scattering. The probable error given for each parameter represents the allowed range of values leading to comparable agreement with experiment.

For elements in the neighbourhood of Al and Fe the charge radii are seen to agree with our results within the estimated probable errors in r_0. The nuclear radius of Pb appears to be significantly larger than its charge radius, a result which may be understood in terms of the neutron excess in that element.

It should be noted that the value of a in the nuclear potential profile is expected to be larger than that deduced from the electron scattering data, since the potential profile represents the folding of a finite-ranged nucleon-nucleon interaction into the actual density distribution. We find in fact that the folding of a Yukawa two-body potential of range 0.8×10^{-13} cm into the electron scattering profile increases a to 0.8×10^{-13} cm and leaves r_0 unchanged, in qualitative agreement with table 1.

4.5. POTENTIAL PARAMETERS

The potential strengths given in table 2 are close to those derived by Bethe [9]) from the nucleon-nucleon scattering amplitudes, except for a discrepancy in the sign of U_2, the imaginary component of the spin-orbit coupling. The values of k_1 and K also compare well with the very approximate results obtained from eq. (4).

TABLE 3

Comparison between measured and calculated values of the total cross section σ and absorption cross section σ_a at 300 MeV. The calculations are based on the parameter assignments in tables 1 and 2

		Al^{27}	Fe^{56}	Pb^{208}
σ_a (b)	Calc.	0.36	0.65	1.75
	Obs.[a])	0.40	0.68	1.70
σ (b)	Calc.	0.52	1.04	3.25
	Obs.[a])	0.57	1.08	3.10

[a]) See ref.[10]).

Table 3 shows the comparison between observed values of σ and σ_a and the values computed from the parameters in tables 1 and 2. The measurements

of σ and σ_a substantially reduce the range over which the parameters of the model may vary if required to agree only with $\sigma(\theta)$ and $P(\theta)$.

We wish to thank Dr. R. Sternheimer for his assistance in setting up the WKB calculation. We are grateful to Mrs. M. Chasson, Mr. H. V. O'Neill and Dr. B. Lepson of the U.S. Naval Research Laboratory, and to Miss I. Stegun and Mr. J. Wegstein of the National Bureau of Standards, for their very able preparation of the cross section programs executed on the NAREC, SEAC, and IBM 704 computers.

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STRANGE PARTICLE CONTRIBUTION TO THE MAGNETIC MOMENT OF THE NUCLEON

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Abstract: The contribution of the strong hyperon–K-meson–nucleon interaction to the nucleon magnetic moment has been calculated, to the first order. In order to fit the observed values with pseudoscalar K-mesons one needs $g_\pi^2/4\pi = 9.6$ and $g_{KNY}^2/g_\pi^2 = 0.06$. If, however, the K-meson is assumed to be scalar, the last ratio becomes 0.016, which is perhaps too small, and $g_\pi^2/4\pi$ in this case is 11.3.

It is well-known that the anomalous magnetic moment of the nucleon could not be explained from the contribution of the pion cloud assuming a pseudoscalar interaction between the mesons and nucleons. In view of the fact that the hyperons and K-mesons interact very strongly with the nucleon the hyperonic and K-mesonic current contribution to the nucleon magnetic moment should also be considered. We, therefore, study the nucleon, hyperon and K-meson fields in presence of an external electromagnetic field. The interaction of the nucleon-hyperon and K-meson field is taken as usual $g_{KNY}\bar{\psi}_N \Gamma \tau_\mu \phi^\mu \psi_K$, with $\Gamma = 1$ or γ_5, and $\phi^\mu = (\Lambda^0, \Sigma)$, $\tau^\mu = (1, \tau)$ $(\mu = 0, 1, 2, 3)$. ψ_N and ψ_K are the nucleon and K-meson fields respectively. The interaction of the electromagnetic field with the K-meson and the hyperon is similar to that with the pion and nucleon fields, respectively. We have performed the calculation assuming either a scalar or a pseudoscalar K-meson and treating all hyperons as fermions of spin $\frac{1}{2}$ and of even parity.

First order diagrams for the π-meson case have been given by many authors, for instance, by Goto [1]) and Fried [2]). The diagrams in which strange particles are emitted are similar except in as much as strangeness and isobaric spin must be conserved at each vertex. Hence only the diagrams corresponding to the following processes are possible:

$$N \leftrightarrow \Sigma^- + K^+, \quad P \leftrightarrow \overset{0}{\Sigma}{}^+ + \overset{+}{K}{}^0, \quad P \leftrightarrow \Lambda^0 + K^+. \tag{1}$$

We assume $g_{\Sigma KN} = g_{\Lambda KN} = g_K$. Following the procedure of ref.[1]) and [2]), the anomalous magnetic moment in the case of the pseudoscalar K-meson is then expressed as

$$\Delta\mu_n = -\frac{1}{\pi}\left(\frac{g_\pi^2}{4\pi}\right)\left[J_F^P + J_B^P - \left(\frac{g_K}{g_\pi}\right)^2 (J_F^\Sigma + J_B^\Sigma)\right], \tag{2}$$

$$\Delta\mu_P = -\frac{1}{\pi}\left(\frac{g_\pi^2}{4\pi}\right)\left[\tfrac{1}{2}J_F^P - J_B^N - \left(\frac{g_K}{g_\pi}\right)^2 (\tfrac{1}{2}J_F^\Sigma - J_B^\Sigma - J_B^\Lambda)\right], \tag{3}$$

where J_F and J_B represent the integrals arising from the fermion and boson currents respectively. The superscripts specify the heavy fermion involved in the diagram. Further,

$$J_F = \tfrac{2}{3}(\rho_F - 1)I_2 + I_3, \tag{4}$$

$$J_B = (\rho_F - 1)I_1 + (2 - \rho_F)I_2 - I_3, \tag{5}$$

$$I_n = \int_0^1 \frac{x^n}{X}\,dx; \quad X = x^2 + bx - \eta_B^2, \tag{6}$$

$$b = \rho_F^2 - 1 - \eta_B^2, \tag{7}$$

where ρ_F, η_B are the ratios of the fermion or boson mass to the nucleon mass. The integrals are evaluated by standard methods and are found to be

$$\begin{aligned} J_F^P &= 0.47, & J_B^P &= J_B^N = 0.34, \\ J_F^\Sigma &= -0.20, & J_B^\Sigma &= 3.38, \\ & & J_B^\Lambda &= 4.13. \end{aligned} \tag{8}$$

Substituting these values in eqs. (2) and (3) we obtain, on solving for the constants,

$$g_\pi^2/4\pi = 9.6, \quad g_K^2/g_\pi^2 = 0.06. \tag{9}$$

These values are in general agreement with other determinations of these constants, although one may think that they are too small. The smallness of the ratio is connected with the relatively large values of the integrals $J_B^{\Sigma,\Lambda}$ in (8). These integrals are, however, quite unambiguously determined.

The important difference between the strange particle integrals and the pion integrals (8) arises from the peculiar way in which the masses of these particles are involved in the corresponding expressions of eq. (4—7). The difference becomes even more pronounced if we consider the K-meson to be a scalar particle. The integrals for this case are obtained from those of the pseudoscalar case by replacing the γ_5 matrices by unity in the well-known way. The magnetic moment may again be expressed by eqs. (2) and (3) with the J's replaced by similar integrals J':

$$J'_F = J_F + \rho_F I_2, \qquad J'_B = J_B + 2\rho_F(I_2 - I_1). \tag{10}$$

The numerical values are

$$J'^\Sigma_F = 5.17, \quad J'^\Sigma_B = 12.2, \quad J'^\Lambda_B = 13.1. \tag{11}$$

To fit the anomalous magnetic moments we now need

$$g_K'^2/g_\pi^2 = 0.016, \quad g_\pi^2/4\pi = 11.3 \tag{12}$$

where $g_K'^2$ is the coupling constant for the K-N-Y interaction with scalar K-mesons. The value for the scalar coupling constant obtained here seems to be rather too small, compared to the value of 0.1 estimated by Mathews and Salam [3]) from scattering data.

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ON THE STRUCTURE OF ROTATIONAL LEVELS OF ATOMIC NUCLEI

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Abstract: It is shown that two kinds of nuclear rotational levels exist. The first kind appears in light nuclei, but a clear rotational energy spectrum is only very seldom connected with this. The second kind appears rather regularly in heavy nuclei with large deformations. There are two other main differences between these two kinds of rotational levels. If in light nuclei a rotational energy spectrum is present, then this spectrum always breaks off abruptly, in contrast to the second kind of rotational spectrum. The second main difference is that the ratios of the γ-transition probabilities for the second kind of rotational levels can be described by the collective model. As opposed to this, the corresponding ratios of the first kind are reduced strongly compared with the values of the collective model. At the end of the paper more general questions are briefly discussed.

1. Introduction

In a preceding paper [1]) (denoted by I) we have discussed a method which is especially useful for investigating the structure of the ground state and of the first excited states of the atomic nuclei. The application of this method to the ground state and the two lowest excited levels of Be^8 has shown that the energies of these levels have approximately the structure of a rotational spectrum. In contrast to the normal collective rotational states of the heavy nuclei which have been investigated particularly by Bohr and Mottelson [2]), the whole energy differences of these states are produced by differences in their potential energy. In all three states the kinetic energy is the same as long as we use an oscillator potential as the average potential for the motion of the nucleons in the nuclei. This indicates that different kinds of rotational states of the nuclei exist.

Therefore it is of interest to investigate if it is possible to divide the rotational states of the nuclei into different limiting cases and to point out what are the essential differences of these limiting cases. We proceed as follows. First we construct the wave functions which we have to assume describe rotational states and we calculate the observable magnitudes which are connected with these states (excitation energies, γ-transition probabilities). Then we discuss the question how far it is possible to derive these states from the nuclear forces. At the end of the paper we consider briefly some special examples and more general questions.

We begin with a re-examination of the lowest energy levels of Be^8. Here

we choose for the description of these states a method which makes the comparison with the normal collective states as simple as possible. Next we extend our considerations to large amplitudes of the relative oscillation of the two α-clusters. Afterwards we pass over to the case of the rotational spectra for even nuclei in general. We shall see that the construction of the rotational wave functions for light and heavy nuclei is the same in principle, but they differ essentially in the associated γ-transition probabilities. Furthermore we shall see that in light nuclei the rotational spectra break off abruptly while in heavy nuclei with large deformations such an abrupt breaking off is not to be expected. Moreover, it can be shown that the appearance of a rotational structure in the energy spectra of light nuclei must be expected to be only sporadic, while in heavy nuclei with large deformations the lowest excitation levels are practically always rotational levels.

2. Rotational States of the Ideal Nucleus Be^8

2.1. THE LOW-ENERGY LEVELS OF Be^8

It was shown in I that the wave functions of the ground state and the first two excited levels of Be^8 are described approximately by

$$\psi = \mathrm{A}\{\psi_0(\alpha_1)\psi_0(\alpha_2)\chi(\mathbf{R}_{\alpha_1}-\mathbf{R}_{\alpha_2})\}; \tag{1}$$

$\psi_0(\alpha_1)$ and $\psi_0(\alpha_2)$ are wave functions of the α-particle in the ground state, $\chi(\mathbf{R}_{\alpha_1}-\mathbf{R}_{\alpha_2})$ describes the relative motion of the two α-particles and is for the ground state a 4s-function, for the first excited state a 4d-function and for the second excited state a 4g-function. The operator A denotes the complete antisymmetrization of the wave function (1) and also normalization of the wave function to unity.

The wave functions (1) can also be constructed from another point of view which shows the close connection of the states (1) to the normal collective rotational states.

For the normal collective rotational states it is always assumed that the deformation of the nucleus in all rotational states is the same and that the states differ only essentially in the angular velocity of the symmetry axis of the nucleus. If we want to construct such rotational states we have to make the following "ansatz" for $\chi(\mathbf{R}_{\alpha_1}-\mathbf{R}_{\alpha_2})$ in (1):

$$\chi(\mathbf{R}_{\alpha_1}-\mathbf{R}_{\alpha_2}) = f(R)\mathrm{e}^{-aR^2}Y_{l,m}(\vartheta,\varphi) \tag{2}$$

$R = |\mathbf{R}_{\alpha_1}-\mathbf{R}_{\alpha_2}|$; ϑ; φ are polar coordinates to $\mathbf{R}_{\alpha_1}-\mathbf{R}_{\alpha_2}$; $a = m\omega/\hbar$ is the width parameter of the oscillator potential; $f(R)$ is the same function for all rotational states. If we choose for $f(R)Y_{l,m}(\vartheta,\varphi)$ the "ansatz"

$$f(R)Y_{l,m}(\vartheta,\varphi) = R^l\mathrm{e}^{-aR^2}Y_{l,m}(\vartheta,\varphi) \tag{3}$$

and introduce this "ansatz" in (1), then after the complete antisymmetriza-

tion we again get the eigenfunctions of the three lowest states of Be^8, if l has the values of 0, 2, 4. For $l = 4$ we get the 4g-function of the oscillator potential. For $l = 2$ and 0 the functions (3) differ from the 4d and 4s functions of the oscillator potential in such a way that terms appear in the oscillator functions which are quadratic in R and others independent of R. These terms give no contributions to the wave functions after antisymmetrization because they correspond to low relative oscillations of the α-particles (see I). As long as we consider the nucleons to be distinguishable and the interaction forces to have no exchange character, the rotational energies are produced only by the change of the kinetic energy of the nucleons, because the relative positions of the nucleons are not changed in the different states. In this case one gets an exact rotational spectrum for any $f(R)$ and any angular momentum values; and, as can easily be seen, the only part of the expectation value of the kinetic energy which depends on the angular momentum has the form

$$E_{\text{rot}} = \frac{\hbar^2 l(l+1)}{Am} 2 \left\langle \frac{1}{R^2} \right\rangle \tag{4}$$

(A = atomic weight, m = mass of a nucleon.)

In the case of Be^8, after antisymmetrization of the wave functions the differences of the kinetic energy between the ground state and the two first excited levels vanish completely. Now the energy differences between these states are produced entirely by the change of the potential energy. The nuclear exchange forces generally enlarge these energy differences †. It further appears that we get a rotational spectrum only if we make for $f(R)$ the above mentioned "ansatz"

$$f(R) = R^4. \tag{5}$$

The rotational spectrum breaks off at $l = 4$, i.e. for $l > 4$ the energy terms are no longer described by the original rotational spectrum. We shall come back to this point later.

2.2. CALCULATION OF THE ENERGY EXPECTATION VALUES

For a more precise investigation of the behaviour described above, of the energy spectrum caused by the transition from distinguishable to indistinguishable particles we use the following representation for the wave functions:

$$\psi = \mathrm{A}\left\{\int g(\mathbf{r}_1 \ldots \mathbf{r}_8)\delta(\mathbf{r}_1-\mathbf{r}'_1)\ldots\delta(\mathbf{r}_8-\mathbf{r}'_8)\mathrm{d}\mathbf{r}'_1\ldots\mathrm{d}\mathbf{r}'_8\right\} \tag{6}$$

$$= \mathrm{A}\left\{\int \psi_0(\alpha'_1)\psi_0(\alpha'_2)f(R')Y_{l,m}(\vartheta',\varphi')\delta(\mathbf{r}_1-\mathbf{r}'_1)\ldots\delta(\mathbf{r}_8-\mathbf{r}'_8)\mathrm{d}\mathbf{r}'_1\ldots\mathrm{d}\mathbf{r}'_8\right\};$$

$\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_8$ are the space, spin and isobaric spin coordinates of the 8

† Therefore both effects produce a velocity dependence of the potential energy.

nucleons. $\delta(\mathbf{r}_1-\mathbf{r}'_1)$, $\delta(\mathbf{r}_2-\mathbf{r}'_2), \ldots \delta(\mathbf{r}_8-\mathbf{r}'_8)$ are eigenfunctions which state that the nucleon 1 has the position $\mathbf{r}'_1$, the spin $+\frac{1}{2}$ and the isobaric spin $+\frac{1}{2}$, the nucleon 2 has the position $\mathbf{r}'_2$, the spin $+\frac{1}{2}$ and the isobaric spin $-\frac{1}{2}$ and so on. As can be seen we obtain the wave functions for the three lowest energy states of Be^8 if we put $f(R') = R'^4$ and $l = 0, 2, 4$.

The parameter representation (6) for the wave functions has the advantage that the antisymmetrization of the wave functions is carried out in the coordinates $\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_8$, but not in the parameter coordinates $\mathbf{r}'_1 \ldots \mathbf{r}'_8$. Therefore we can introduce any other parameter coordinates for the parameter $\mathbf{r}'_1 \ldots \mathbf{r}'_8$ without being cautious about antisymmetrization prescriptions. It is, for instance, very practical to introduce parameter coordinates which describe the motion of the nucleons in relation to the centre of mass coordinates of the two α-particles and which describe the relative motion of the centre of mass coordinates of the two α-particles in spherical coordinates. In (6) this is signified by the coordinates α'_1, α'_2, R', ϑ', φ'.

For the expectation value of any operator $O(1 \ldots 8)$ which depends on the 8 nucleons we obtain

$$\begin{aligned} N^2\langle O\rangle &= \int [\mathrm{A}\psi^*(\mathbf{r}_1 \ldots \mathbf{r}_8) O(1 \ldots 8) \mathrm{A}\psi(\mathbf{r}_1 \ldots \mathbf{r}_8)]\, \mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_8 \\ &= 8! \int [\mathrm{A}\psi^*(\mathbf{r}_1 \ldots \mathbf{r}_8) O(1 \ldots 8) \psi(\mathbf{r}_1 \ldots \mathbf{r}_8)] \mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_8 \\ &= 8! \int [\mathrm{A}\{g^*(\mathbf{r}''_1 \ldots \mathbf{r}''_8)\delta(\mathbf{r}_1-\mathbf{r}''_1) \ldots \delta(\mathbf{r}_8-\mathbf{r}''_8)] \mathrm{d}\mathbf{r}''_1 \ldots \mathrm{d}\mathbf{r}''_8\} \\ &\quad O(1 \ldots 8) \left\{ \int g(\mathbf{r}'_1 \ldots \mathbf{r}'_8)\delta(\mathbf{r}_1-\mathbf{r}'_1) \ldots \delta(\mathbf{r}_8-\mathbf{r}'_8) \mathrm{d}\mathbf{r}'_1 \ldots \mathrm{d}\mathbf{r}'_8 \right\} \Big] \mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_8. \end{aligned} \tag{7}$$

(N is the normalization factor of the wave function.) The integrations in (7) mean at the same time summation over the corresponding spin and isobaric spin coordinates.

If we assume for the present that the operator $O(1 \ldots 8)$ does not describe any spin and isobaric spin exchange, then it can be seen easily that — besides the term with no exchange — only such terms contribute to the expectation value (7) in which only those nucleons are exchanged which are in the same spin and isobaric spin states †. Furthermore it can be seen that due to the high symmetry of the Be^8-wave functions for even l all one-pair and three-pair exchange terms give the same contribution to (7). The same is valid for the direct term (no exchange) and for the four-pair exchange term. All two-pair exchange terms are also equal to one another. If l is odd, the antisymmetrization always gives 0. This describes the fact that the α-particles are Bose particles (see also I).

† I.e. we have — besides the term with no exchange — only terms in which particle 1 is exchanged with particle 5, particle 2 with particle 6, and so on.

We therefore obtain from (7) for even l

$$\begin{aligned} N^2\langle O\rangle &= 2\cdot 8!\int[\{\psi^*(\mathbf{r}_1\ldots\mathbf{r}_8)-4\psi^*(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_8,\mathbf{r}_5,\mathbf{r}_6,\mathbf{r}_7,\mathbf{r}_4)\\ &\quad+3\psi^*(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_7,\mathbf{r}_8,\mathbf{r}_5,\mathbf{r}_6,\mathbf{r}_3,\mathbf{r}_4)\}O(1\ldots 8)\psi(\mathbf{r}_1\ldots\mathbf{r}_8)\mathrm{d}\mathbf{r}_1\ldots\mathrm{d}\mathbf{r}_8\\ &= 2\cdot 8!\int\Big[\int g(\mathbf{r}''_1\ldots\mathbf{r}''_8)\{\delta(\mathbf{r}_1-\mathbf{r}''_1)\ldots\delta(\mathbf{r}_8-\mathbf{r}''_8)\\ &\quad-4\delta(\mathbf{r}_1-\mathbf{r}''_1)\ldots\delta(\mathbf{r}_8-\mathbf{r}''_4)\delta(\mathbf{r}_5-\mathbf{r}''_5)\ldots\delta(\mathbf{r}_4-\mathbf{r}''_8)\\ &\quad+3\delta(\mathbf{r}_1-\mathbf{r}''_1)\ldots\delta(\mathbf{r}_7-\mathbf{r}''_3)\delta(\mathbf{r}_8-\mathbf{r}''_4)\ldots\delta(\mathbf{r}_3-\mathbf{r}''_7)\delta(\mathbf{r}_4-\mathbf{r}''_8)\}\\ &\mathrm{d}\mathbf{r}''_1\ldots\mathrm{d}\mathbf{r}''_8 O(1\ldots 8)\int\{g(\mathbf{r}'_1\ldots\mathbf{r}'_8)\delta(\mathbf{r}_1-\mathbf{r}'_1)\ldots\\ &\qquad\qquad\delta(\mathbf{r}_8-\mathbf{r}'_8)\mathrm{d}\mathbf{r}'_1\ldots\mathrm{d}\mathbf{r}'_8\}\Big]\mathrm{d}\mathbf{r}_1\ldots\mathrm{d}\mathbf{r}_8. \end{aligned} \tag{8}$$

First we calculate the expectation value of the potential energy. The operator $O(1\ldots 8)$ is in this special case a superposition of two particle operators $\mathscr{V}(\mathbf{r}_1-\mathbf{r}_k)$. For the sake of simplicity we assume that the two-particle interactions are δ-interactions. Therefore $O(1\ldots 8)$ here has the form

$$O(1\ldots 8) = -\tfrac{1}{2}\mathscr{V}\sum_{\substack{ik\\ i\neq k}}\delta(\mathbf{r}_i-\mathbf{r}_k). \tag{9}$$

Instead of the parameter coordinates $\mathbf{r}'_1\ldots\mathbf{r}'_8$ we introduce the following centre of mass and relative coordinates:

$$\mathbf{R}'_1=\tfrac{1}{4}(\mathbf{r}'_1+\mathbf{r}'_2+\mathbf{r}'_3+\mathbf{r}'_4),\quad \mathbf{R}'_2=\tfrac{1}{4}(\mathbf{r}'_5+\mathbf{r}'_6+\mathbf{r}'_7+\mathbf{r}'_8)$$
$$\bar{\mathbf{r}}'_2=\mathbf{r}'_2-\mathbf{R}'_1,\qquad \bar{\mathbf{r}}'_6=\mathbf{r}'_6-\mathbf{R}'_2,$$
$$\bar{\mathbf{r}}'_3=\mathbf{r}'_3-\mathbf{R}'_1,\qquad \bar{\mathbf{r}}'_7=\mathbf{r}'_7-\mathbf{R}'_2,$$
$$\bar{\mathbf{r}}'_4=\mathbf{r}'_4-\mathbf{R}'_1,\qquad \bar{\mathbf{r}}'_8=\mathbf{r}'_8-\mathbf{R}'_2.$$

A corresponding substitution is made for the parameter coordinates $\mathbf{r}''_1\ldots\mathbf{r}''_8$.

If we carry out all integrations in (6), except over the directions of $\mathbf{R}'_1-\mathbf{R}'_2$, with these new parameter coordinates, then the wave function (6) is represented as a superposition of completely antisymmetrized (internal) wave functions ψ_{int}, in which ϑ' may assume all values between 0 and π and φ' may assume all values between 0 and 2π. Thus we obtain for this wave function the general form

$$\psi=\int\psi_{\mathrm{int}}(\mathbf{r}_i)Y_{l,m}(\vartheta',\varphi')\mathrm{d}\,\cos\vartheta'\mathrm{d}\varphi'. \tag{6a}$$

At this point the relation of our method for the construction of rotational wave functions to the method of Yoccoz and Peierls [3]) can already be seen. But in contrast to Yoccoz and Peierls we can always split off the centre of mass coordinates in a clear way. We are not restricted to oscillator wave functions. We can take any other wave functions as wave functions for the nucleons inside the clusters and for the relative oscillations of the clusters

without violating the conservation laws of momentum and angular momentum. We shall come back to these points later.

After this interruption we continue the calculation of the expectation value of the potential energy. We put for $g(\mathbf{r}'_1 \ldots \mathbf{r}'_8)$ (see eqs. (6) and (7))

$$g(\mathbf{r}'_1 \ldots \mathbf{r}'_8) = \exp\left[-\tfrac{1}{2}a\sum_{i=1}^{4}\bar{\mathbf{r}}_i'^2 - \tfrac{1}{2}a\sum_{i=5}^{8}\mathbf{r}_i'^2\right] R'^n \mathrm{e}^{-aR'^2} Y_{l,m}(\vartheta', \varphi'), \quad (11)$$

with

$$\mathbf{r}'_1 = \mathbf{r}'_1 - \mathbf{R}'_1 = -(\bar{\mathbf{r}}'_2 + \bar{\mathbf{r}}'_3 + \bar{\mathbf{r}}'_4),\ \bar{\mathbf{r}}'_5 = -(\bar{\mathbf{r}}'_6 + \bar{\mathbf{r}}'_7 + \bar{\mathbf{r}}'_8),\ \mathbf{R}' = \mathbf{R}'_1 - \mathbf{R}'_2.$$

With this g-function the wave function (6) describes a state of oscillation of two α-clusters against each other, in which the relative oscillation is described by $R'^n Y_{l,m}(\vartheta', \varphi')\mathrm{e}^{-aR'^2}$. If we choose n small or large, we get a relative oscillation with relatively small or large amplitude.

If we introduce (11) in (8) and integrate over all coordinates of the nucleons and over all parameter coordinates (10) except the relative parameter coordinates $\mathbf{R}'$ and $\mathbf{R}''$, we obtain after somewhat tiresome but not difficult integrations:

$$\langle O\rangle = \langle \mathscr{V}\rangle = -\mathscr{V}\cdot 3\cdot 8\cdot 8!\,\frac{B_1+B_2-2B_3-B_4-4B_5+B_6+4B_7}{N^2} \quad (12a)$$

with the direct terms

$$\begin{aligned} B_1 &= \frac{\pi^9}{a^9}\int R'^{2(n+1)}\mathrm{e}^{-2aR'^2}|Y_{l,m}(\vartheta', \varphi')|^2 \mathrm{d}R'\,\mathrm{d}\Omega' \\ B_2 &= 2^3\left(\frac{\pi^6}{3a^6}\right)^{\frac{3}{2}} R'^{2(n+1)}\mathrm{e}^{-(8/3)aR'^2}|Y_{l,m}(\vartheta', \varphi')|^2 \mathrm{d}R'\mathrm{d}\Omega' \end{aligned} \quad (12b)$$

and the exchange terms

$$\begin{aligned} B_3 &= 2^6\left(\frac{\pi^5}{6a^5}\right)^{\frac{3}{2}}\int R''^{n+2}R'^{n+2}\mathrm{e}^{-(8/3)a(R''^2-\mathbf{R}''\cdot\mathbf{R}'+R'^2)} \\ &\qquad \cdot Y^*_{l,m}(\vartheta'', \varphi'')Y_{l,m}(\vartheta', \varphi')\mathrm{d}R''\mathrm{d}R'\mathrm{d}\Omega''\mathrm{d}\Omega' \\ B_4 &= 2^3\left(\frac{\pi^5}{a^5}\right)^{\frac{3}{2}}\int R''^{n+2}R'^{n+2}\mathrm{e}^{-a(3R''^2-2\mathbf{R}''\cdot\mathbf{R}'+3R'^2)} \\ &\qquad \cdot Y^*_{l,m}(\vartheta'', \varphi'')Y_{l,m}(\vartheta', \varphi')\mathrm{d}R''\mathrm{d}R'\mathrm{d}\Omega''\mathrm{d}\Omega' \\ B_5 &= 2^3\left(\frac{\pi^5}{a^5}\right)^{\frac{3}{2}}\int R''^{n+2}R'^{n+2}\mathrm{e}^{-a(3R''^2-4\mathbf{R}''\cdot\mathbf{R}'+4R'^2)} \\ &\qquad \cdot Y^*_{l,m}(\vartheta'', \varphi'')Y_{l,m}(\vartheta', \varphi')\mathrm{d}R''\mathrm{d}R'\mathrm{d}\Omega''\mathrm{d}\Omega' \\ B_6 &= 2^3\left(\frac{\pi^5}{2a^5}\right)^{\frac{3}{2}}\int R''^{n+2}R'^{n+2}\mathrm{e}^{-2a(R''^2+R'^2)} \\ &\qquad \cdot Y^*_{l,m}(\vartheta'', \varphi'')Y_{l,m}(\vartheta', \varphi')\mathrm{d}R''\mathrm{d}R'\mathrm{d}\Omega''\mathrm{d}\Omega' \\ B_7 &= 2^6\left(\frac{\pi^5}{6a^5}\right)^{\frac{3}{2}}\int R''^{n+2}R'^{n+2}\mathrm{e}^{-a((8/3)R''^2+2R'^2)} \\ &\qquad \cdot Y^*_{l,m}(\vartheta'', \varphi'')Y_{l,m}(\vartheta', \varphi')\mathrm{d}R''\mathrm{d}R'\mathrm{d}\Omega''\mathrm{d}\Omega', \end{aligned} \quad (12c)$$

and †

$$N^2 = 2 \cdot 8! \left(\frac{2\pi}{a}\right)^3 (B_1 - 4B_3 + 3B_6). \tag{12d}$$

In the calculation of $\langle O \rangle$ we have already collected all the terms which are equal to each other after the integration over R'', ϑ'', φ'' and R', ϑ', φ'. That is the reason why (12) is not completely symmetrical in R'', ϑ'', φ'' and R', ϑ', φ' as one would expect.

If in the relations (12) we fix the directions ϑ'', φ'' and ϑ', φ', we replace the spherical harmonics and N^2 by 1 and carry out only the integrations over R' and R'', then we get from the relations (12) the transition matrix element of the potential energy between two states in which the two α-clusters in the first state oscillate in the direction ϑ', φ' and in the second state in the direction ϑ'', φ''. The amplitudes of oscillation are the same in both states. We denote such a transition matrix element of the potential energy by †† $\langle \vartheta'', \varphi'' | \mathscr{V} | \vartheta', \varphi' \rangle$.

Similarly to the expectation value of the potential energy we can obtain the expectation value of the kinetic energy operator

$$T = -\sum_{i=1}^{8} \frac{\hbar^2}{2m} \Delta_i \tag{13}$$

(m = mass of a nucleon). The result is

$$\begin{aligned} \langle T \rangle &= \frac{\hbar^2}{2m} a(n+12) - 8! \frac{\hbar^2}{2m} \frac{n(n+1) - l(l+1)}{2N^2} \int (\mathrm{A}\psi^*) \frac{1}{R^2} \psi \mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_8 \\ &= \frac{\hbar^2}{2m} \left\{ a(n+12) - 8! \frac{n(n+1) - l(l+1)}{2} \frac{C_1 - C_2 + C_3}{N^2} \right\} \end{aligned} \tag{14a}$$

with the direct term

$$C_1 = 2^7 \left(\frac{\pi^7}{8a^7}\right)^{\frac{1}{2}} \int R'^{2n} \mathrm{e}^{-2aR'^2} |Y_{l,m}(\vartheta', \varphi')|^2 \mathrm{d}R' \mathrm{d}\Omega' \tag{14b}$$

and the exchange terms

$$\begin{aligned} C_2 &= 2^9 \left(\frac{\pi^6}{3a^6}\right)^{\frac{1}{2}} \int R''^n R'^{n+2} \mathrm{e}^{-(8/3)a(R''^2 - \mathbf{R}'' \cdot \mathbf{R}' + R'^2)} Y^*_{l,m}(\vartheta'', \varphi'') Y_{l,m}(\vartheta', \varphi') \\ &\qquad \cdot \mathrm{d}R'' \mathrm{d}R' \mathrm{d}\Omega'' \mathrm{d}\Omega', \\ C_3 &= 3 \cdot 2^4 \left(\frac{\pi^6}{a^6}\right)^{\frac{1}{2}} \int R''^n R'^{n+2} \mathrm{e}^{-2a(R''^2 + R'^2)} Y_{l,m}(\vartheta'', \varphi'') Y_{l,m}(\vartheta', \varphi') \\ &\qquad \cdot \mathrm{d}R'' \mathrm{d}R' \mathrm{d}\Omega'' \mathrm{d}\Omega'. \end{aligned} \tag{14c}$$

The transition matrix element of the kinetic energy for the directions ϑ', φ' and ϑ'', φ'' which is defined in a manner completely analogous to the

† To get N^2 from the eqs. (7) and (8) we set $O(1 \ldots 8)$ simply = 1.

†† The parts of $\langle \vartheta'', \varphi'' | \mathscr{V} | \vartheta', \varphi' \rangle$ which come from the direct terms B_1 and B_2 have the behaviour of a δ-function due to the variable γ.

transition matrix element of the potential energy, is denoted by $\langle \vartheta'', \varphi'' | T | \vartheta', \varphi' \rangle$.

As can be seen, these transition matrix elements of the potential and kinetic energy depend on the angle γ between the two directions ϑ'', φ'' and ϑ', φ' only and not on the choice of the special coordinate system. This also follows from general invariance arguments.

If we carry the remaining integrations in (12) and (14), we obtain for the quantities B_i and C_i:

$$B_1 = \sqrt{\frac{\pi}{2a}} \frac{\pi^9}{a^{n+10}} \frac{(2n+1)!!}{2^{2n+3}}$$

$$B_2 = \sqrt{\frac{\pi}{2a}} \frac{\pi^9}{a^{n+10}} \frac{(2n+1)!!}{2^{2n+3}} (\tfrac{3}{4})^n$$

$$B_3 = \sqrt{\frac{\pi}{2a}} \frac{\pi^9}{a^{n+10}} \frac{1}{2^{2n+l+3}} \frac{[(n+l+1)!!]^2}{(2l+1)!!} {}_2F_1(-\tfrac{1}{2}(n-l), -\tfrac{1}{2}(n-l); l+\tfrac{3}{2}; \tfrac{1}{4})$$

$$B_4 = \sqrt{\frac{\pi}{2a}} \frac{\pi^9}{a^{n+10}} \frac{3^{n-l}}{2^{4n+3}} \frac{[(n+l+1)!!]^2}{(2l+1)!!} {}_2F_1(-\tfrac{1}{2}(n-l), -\tfrac{1}{2}(n-l); l+\tfrac{3}{2}; \tfrac{1}{9}) \quad (15)$$

$$B_5 = \sqrt{\frac{\pi}{2a}} \frac{\pi^9}{a^{n+10}} \frac{3^{\frac{1}{2}(n-l)}}{2^{3n+3}} \frac{[(n+l+1)!!]^2}{(2l+1)!!} {}_2F_1(-\tfrac{1}{2}(n-l), -\tfrac{1}{2}(n-l); l+\tfrac{3}{2}; \tfrac{1}{3})$$

$$B_6 = \sqrt{\frac{\pi}{2a}} \frac{\pi^9}{a^{n+10}} \frac{1}{2^{2n+3}} [(n+1)!!]^2 \delta_{l0}$$

$$B_7 = \sqrt{\frac{\pi}{2a}} \frac{\pi^9}{a^{n+10}} \frac{3^{\frac{1}{2}n}}{2^{3n+3}} [(n+1)!!]^2 \delta_{l0};$$

$$C_1 = \frac{\pi^{11}}{a^{n+11}} \frac{(2n-1)!!}{2^{2n-1}}$$

$$C_2 = -\frac{\pi^{11}}{a^{n+11}} \frac{(2n-1)!!}{2^{2n-l-3}} {}_2F_1(-\tfrac{1}{2}(n-l), -\tfrac{1}{2}(n-l-2); -\tfrac{1}{2}(2n-1); \tfrac{3}{4}) \quad (16)$$

$$C_3 = \frac{\pi^{11}}{a^{n+11}} \frac{3(n+1)[(n-1)!!]^2}{2^{2n-1}} \delta_{l0}.$$

In (15) and (16) $k!! = 1 \cdot 3 \ldots k$ for odd k and $k!! = 2 \cdot 4 \ldots k$ for even k; ${}_2F_1(a, b; c; d)$ is the hypergeometric function of the indices 2,1 and the arguments $a, b; c; d$ (see ref.[4]) pages 10 ff. and ref.[5]) page 280 ff.). Further we have used the expansion (see for instance ref.[4]) p. 28 ff.)

$$\begin{aligned} e^{-\lambda \mathbf{R}'' \cdot \mathbf{R}'} &= e^{-\lambda R'' R' \cos\gamma} \\ &= (4\pi[2l+1])^{\frac{1}{2}} \sum_{l=0}^{\infty} (-1)^l \sqrt{\frac{\pi}{2\lambda R'' R'}} I_{l+\frac{1}{2}}(\lambda R'' R') Y_{l,0}(\gamma) \\ &= 4\pi \sum_{l=0}^{\infty} (-1)^l \sqrt{\frac{\pi}{2\lambda R'' R'}} I_{l+\frac{1}{2}}(\lambda R'' R') Y^*_{l,m}(\vartheta'', \varphi'') Y_{l,m}(\vartheta', \varphi'); \end{aligned} \quad (17)$$

$I_{l+\frac{1}{2}}(\lambda R'' R')$ are the modified cylinder functions with half-integral indices of the imaginary argument $i\lambda R'' R'$.

2.3. STRUCTURE OF THE ROTATIONAL LEVELS FOR SMALL AND LARGE DEFORMATIONS OF THE IDEAL NUCLEUS

Here we use the expressions (12), (14), (15) and (16) for the discussion of the energy spectrum of two α-clusters rotating around each other with small or large separation.

At first we consider the cases $n = 4$, $l = 0, 2, 4$, i.e. those which correspond to the ground state and the two first excited levels of Be^8. As it can be seen from (14) and (16) all 3 states have the same kinetic energy, because in all 3 states the terms of the kinetic energy (14) which depend on the angular momenta vanish: for $l = 0$ and $l = 2$ we have $C_1 - C_2 + C_3 = 0$ and for $l = 4$, we have $n(n+1) - l(l+1) = 0$. In all three cases ${}_2F_1(a, b; c; d)$ breaks off, because a and b are 0 or negative integers (see for instance ref.[4]) p. 10). For the potential energies of these three states we get from (12) and (15)

$$E_{n=4,\, l=0} = -\mathscr{V}\left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{70}{2^5} \quad \text{(4s-state)}$$

$$E_{n=4,\, l=2} = -\mathscr{V}\left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{67}{2^5} \quad \text{(4d-state)} \tag{18}$$

$$E_{n=4,\, l=4} = -\mathscr{V}\left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \frac{60}{2^5} \quad \text{(4g-state).}$$

As in I we get †

$$\frac{E_g - E_s}{E_d - E_s} = \frac{10}{3}.$$

If we go over to larger l-values ($l > 4$), then a and b in ${}_2F_1(a, b; c; d)$ become positive, and the hypergeometric function no longer breaks off. From this it follows for instance that the terms in (14) which depend on the angular momentum no longer vanish. This changes the kinetic energy of these states compared with the states for which l is equal to or smaller than 4. From this alone it follows already that the energies of these states are no longer incorporated in the original rotational spectrum. Also the potential energies themselves no longer are incorporated in the original rotational spectrum for $l > 4$ as can be seen without further explanation.

From the representation which we have used here it can be seen very clearly that the breaking off of the rotational spectrum is caused by the exchange terms $B_3 \ldots B_7$, C_2, C_3 in (12), (14), (15) and (16). Without

† *Note added in proof:* Prof. Feenberg has kindly informed us that he and Miss Phillips have also derived the $l(l+1)$-structure of the energy levels for nucleons in the p-shell and LS-coupling [21]).

exchange terms, i.e. considering the nucleons as distinguishable, the rotational spectrum would not break off for $l > 4$ for the given wave functions. Further this representation shows that the change of the rotational energies from kinetic to potential energy with the transition from distinguishable to indistinguishable particles is also caused by these exchange terms.

Now we consider the limiting case of very large amplitude of the relative oscillation of the two α-clusters, i.e. we choose n very large. To this end it is very useful to investigate directly the asymptotic behaviour of the integrals (12a) and (14a) for large n and not to start from the asymptotic behaviour of the hypergeometric function for large n. In this way it is possible to get a clearer insight into how the rotational spectra of nuclei with large deformations come about. The connection with the method of Yoccoz and Peierls [3]) also becomes especially clear.

All expressions in (12) and (14) which cannot be computed explicitly for any n contain the following general integrals:

$$\int e^{+(\lambda_1 r_1^2+\lambda_2 r_2^2)+\beta r_1 r_2 \cos\gamma} r_1^n r_2^n Y^*_{l,m}(1) Y_{l,m}(2) \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \qquad (20)$$

and

$$\int e^{-\lambda(r_1^2+r_2^2)+\beta r_1 r_2 \cos\gamma} r_1^{n+2} r_2^n Y^*_{l,m}(1) Y_{l,m}(2) \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}\Omega_1 \mathrm{d}\Omega_2. \qquad (21)$$

First we consider the integral (20). For large n the integrations over r_1 and r_2 can be carried out by the saddle point method, if one introduces the new integration variables $w = r_1 + r_2\sqrt{(\lambda_2/\lambda_1)}$ and $z = r_1 - r_2\sqrt{(\lambda_2/\lambda_1)}$. One gets

$$\int e^{-\lambda(r_1^2+r_2^2)+\beta r_1 r_2 \cos\gamma} r_1^n r_2^n Y^*_{l,m}(1) Y_{l,m}(2) \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}\Omega_1 \mathrm{d}\Omega_2$$

$$= \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}(n+1)} \frac{\pi}{\sqrt{2\lambda_1}} \left(\frac{n}{e}\right)^n \frac{1}{(2\lambda_1 - \beta\sqrt{(\lambda_1/\lambda_2)})^{\frac{1}{2}(2n+1)}} \int e^{-\frac{1}{2}(2n+1)\ln(2\lambda_1 - \beta\sqrt{(\lambda_1/\lambda_2)}\cos\gamma)} Y^*_{l,m}(1) Y_{l,m}(2) \mathrm{d}\Omega_1 \mathrm{d}\Omega_2. \qquad (22)$$

For the angular integrations it is convenient to expand

$$g(\gamma) = e^{-\frac{1}{2}(2n+1)\ln(2\lambda_1 - \beta\sqrt{(\lambda_1/\lambda_2)}\cos\gamma)} \qquad (23)$$

in spherical harmonics $Y_{l,0}(\cos\gamma)$.

With the help of the addition theorem of the spherical harmonics it is possible to represent $Y_{l,0}(\cos\gamma)$ by a sum over products of spherical harmonics, which depend on the angles ϑ_1, φ_1 and ϑ_2, φ_2. Therefore we set

$$g(\gamma) = e^{-\frac{1}{2}(2n+1)\ln(2\lambda_1 - \beta\sqrt{(\lambda_1/\lambda_2)}\cos\gamma)} = \sum g_l Y_{l,0}(\cos\gamma) \qquad (24)$$

with

$$g_l = 2\pi \int e^{-\frac{1}{2}(2n+1)\ln(2\lambda_1 - \beta\sqrt{\lambda_1/\lambda_2}\cos\gamma)} Y^*_{l,0}(\cos\gamma) \mathrm{d}\cos\gamma \qquad (24a)$$

and

$$Y_{l,0}(\cos\gamma) = \frac{2\sqrt{\pi}}{\sqrt{2l+1}} \sum_{m=-l}^{m=+l} Y_{l,m}(\vartheta_1, \varphi_1) Y^*_{l,m}(\vartheta_2, \varphi_2). \qquad (25)$$

With this we obtain for the integral (22)

$$\int e^{-\frac{1}{2}(2n+1)\ln(2\lambda_1-\beta\sqrt{(\lambda_1/\lambda_2)}\cos\gamma)} Y^*_{l,m}(\vartheta_1,\varphi_1) Y_{l,m}(\vartheta_2,\varphi_2)\,d\Omega_1\,d\Omega_2$$

$$= \frac{2\sqrt{\pi}\,2\pi}{\sqrt{2l+1}} \int e^{-\frac{1}{2}(2n+1)\ln(2\lambda_1-\beta\sqrt{(\lambda_1/\lambda_2)}\cos\gamma)} Y^*_{l,0}(\cos\gamma)\sin\gamma\,d\gamma. \tag{26}$$

For large n values only very small γ-values contribute to the integral (26). Therefore we can expand $Y_{l,0}(\cos\gamma)\sin\gamma$ in powers of γ and break off after the term of the third power in γ. Because of †

$$Y^*_{l,0}(\cos\gamma)\sin\gamma \approx \tfrac{1}{2}\sqrt{\frac{2l+1}{\pi}}\,(1-l(l+1)\gamma^2)\gamma \tag{27}$$

we then finally obtain

$$\int e^{-(\lambda_1 r_1^2+\lambda_2 r_2^2)+\beta r_1 r_2\cos\gamma} r_1^n r_2^n Y^*_{l,m}(\vartheta_1,\varphi_1) Y_{l,m}(\vartheta_2,\varphi_2)\,dr_1\,dr_2\,d\Omega_1\,d\Omega_2 \tag{28}$$

$$\approx \frac{2\sqrt{2}\pi^2}{\sqrt{\lambda_1}} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}n} \left(\frac{n}{e}\right)^n \frac{1}{2n+1} \frac{1}{\beta\left(2\lambda_1-\beta\sqrt{(\lambda_1/\lambda_2)}\right)^{\frac{1}{2}(2n-1)}} \left\{1-l(l+1)\,\frac{2\lambda_1-\beta\sqrt{(\lambda_1/\lambda_2)}}{(2n+1)\beta\sqrt{(\lambda_1/\lambda_2)}}\right\}.$$

We obtain the integral (21) if we set $\lambda_1 = \lambda_2 = \lambda$, differentiate with respect to λ, and multiply the result by a factor $-\frac{1}{2}$. We get

$$\int e^{-\lambda(r_1^2+r_2^2)+\beta r_1 r_2\cos\gamma} r_1^{n+2} r_2^n Y^*_{l,m}(\vartheta_1,\varphi_1) Y_{l,m}(\vartheta_2,\varphi_2)\,dr_1\,dr_2\,d\Omega_1\,d\Omega_2$$

$$\approx \sqrt{2}\,\frac{\pi^2}{\lambda^{\frac{1}{2}}} \left(\frac{n}{e}\right)^n \frac{1}{2n+1} \frac{4n\lambda-\beta}{\beta(2\lambda-\beta)^{\frac{1}{2}(2n+1)}} \left\{1-l(l+1)\,\frac{2\lambda-\beta}{(2n+1)\beta}\right\}. \tag{29}$$

If we calculate with the formulas (28) and (29) the B_i and C_i we obtain for the largest terms

$$\begin{aligned}
B_1 &\approx \frac{\pi^{\frac{1}{2}}}{a^{\frac{1}{2}(2n+21)}} \left(\frac{n}{e}\right)^n \frac{1}{2^n}\,\frac{n}{4} \\
B_2 &\approx \frac{\pi^{\frac{1}{2}}}{a^{\frac{1}{2}(2n+21)}} \left(\frac{n}{e}\right)^n \frac{1}{2^n}\,\frac{n}{4}\,(\tfrac{3}{4})^n \\
B_3 &\approx \frac{\pi^{\frac{1}{2}}}{a^{\frac{1}{2}(2n+21)}} \left(\frac{n}{e}\right)^n \frac{1}{2^n}\,\frac{n^2}{2n+5}\,(\tfrac{3}{4})^n (\tfrac{3}{4})^{\frac{1}{2}} \left(1-\frac{l(l+1)}{2n}\right); \\
C_1 &\approx \frac{\pi^{11}}{a^{n+11}} \left(\frac{n}{e}\right)^n \frac{1}{2^n}\,2^{\frac{1}{2}} \\
C_2 &\approx \frac{\pi^{11}}{a^{n+11}} \left(\frac{n}{e}\right)^n \frac{1}{2^n}\,(\tfrac{3}{4})^n\,\frac{4n-1}{2n+1}\,2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}} \left(1-\frac{l(l+1)}{2n}\right).
\end{aligned} \tag{30}$$

For the calculation of the relations (30) moreover the asymptotic relation $(n-1)!! \sim \sqrt{2}(n/e)^{\frac{1}{2}n}$ for $n \gg 1$ was used. All other terms are smaller at least by a factor $S^n (S < 1)$. Therefore for large n values, i.e. large amplitudes

† *Note added in proof:* In eq. (27) and the following ones, we have forgotten a term proportional to γ^3: but this term does not affect the angular momentum dependence of the energy. Therefore, our conclusions are not influenced by this omission.

of the relative oscillation, these terms can be neglected. This is especially valid for the terms which do not depend on γ. The general reason why for large oscillation amplitudes these terms become small compared with the other terms is that in this case two internal wave functions ψ_{int} corresponding to two different directions overlap considerably only if γ is very small. This means that their associated oscillation directions are almost the same. We shall come back to this again. By means of the relations (12), (14) and (30) one can calculate the expectation value of the total energy

$$\langle E\rangle = \langle \mathcal{V}\rangle + \langle T\rangle. \tag{31}$$

As can be seen immediately for a given radial oscillation we again obtain a rotational spectrum as long as l is much smaller than n. In this case the total energy can be expanded in powers of $l(l+1)$ and to a good approximation can be broken off after the first expansion term. In contrast to the rotational spectrum for small amplitudes this rotational spectrum does not break off abruptly. The appearance of a rotational spectrum for large amplitudes does not require such special conditions as for small amplitudes. Rotational spectra for small amplitudes, which appear only sporadically, will be called rotational spectra of the first kind, while the other will be called rotational spectra of the second kind. As can be seen from this derivation the rotational spectra of the second kind arise from the fact that two internal wave functions ψ_{int} have a big overlap only if the two oscillations directions are very near to each other. This is exactly the same reason which is responsible for the rotational spectra in the method of Yoccoz and Peierls. Also similarly to the results of Yoccoz and Peierls, for our given wave functions the deviations from the rotational spectra become larger and larger with increasing l.

It has to be mentioned that in the special model which we have considered till now, for large n the contributions of the exchange terms B_3 and C_3 to the rotational energies can also be neglected in comparison with the contribution of the direct term of the kinetic energy C_1. For instance, the contribution of B_3 to the rotational energies is smaller by a factor of the order of magnitude $(\frac{3}{4})^n$ than the contribution of C_1. This comes from the fact that in our ideal nucleus the number of nucleons in the clusters does not increase with increasing amplitude of their relative oscillation. From this it follows that the mutual penetration of the two clusters becomes smaller and smaller as this amplitude increases. The contribution of the exchange terms to the rotational energies therefore becomes more and more negligible with increasing amplitude compared with the contribution of the direct term of the kinetic energy.

However, what we want to mention here is that if we consider real nuclei and pass from smaller to larger nuclei, then with increasing nuclear deformation the number of nucleons also increases. Here this means increasing

amplitude of the relative oscillation of the clusters as well as an increasing number of nucleons in the clusters. The mutual penetration of the clusters then remains essentially the same. Therefore the exchange terms remain of the same order of magnitude as the direct terms.

For large amplitudes of the relative oscillation we obtain by means of eqs. (12), (14) and (30) for the energy distances of the rotational levels from the ground level

$$\Delta E_{l,m} \approx \frac{a}{2mn} \hbar^2 l(l+1). \tag{32}$$

This is to be expected, because the wave function of the radial oscillation $R'^n e^{-aR'^2}$ means that the two α-clusters are oscillating around their mean distance $\sqrt{(n/2a)}$ with an amplitude of the order of magnitude $1/\sqrt{a}$, and if $\sqrt{(n/a)} \gg 1/\sqrt{a}$ the rotational spectrum (32) also follows directly from a correspondence consideration.

Until now in all our considerations we have not assumed any exchange forces, neither have we taken into account the finite range of the nuclear forces †: the exchange terms in (12), (14), (16) and (30) come only from the antisymmetrization of the wave function.

It is possible to take into account the exchange character and the finite range of the nuclear forces very generally, for instance by expanding the nuclear forces in Gaussian functions. Then in the potential energy terms the same integrals as in the formulas (12) appear, only with changed constants. It follows that the qualitative behaviour of the new terms is the same as that of the original ones. Only the quantitative results change. Therefore, all preceding arguments about the structure of the rotational levels and their excitation energies remain valid. Thus, it was already shown in I that we obtain for the first three levels of Be^8 a rotational spectrum with a general (two-body-force) "ansatz" for the nuclear forces. Furthermore we found that with a Serber "ansatz" for these forces, which describes the low energy two-nucleon scattering data we obtain the absolute values of the energy of these three states to about 10—20 %.

3. Rotational States of General Even Nuclei with a Symmetry Axis

Now we apply our method to the rotational states of general even nuclei having a symmetry axis. Further, we assume that we have a complete cancellation of the nucleon spins: this is true for practically all even nuclei. If the cluster structure of such a nucleus is given, it must be possible to group the clusters in larger clusters, until finally we are left with only two clusters. The line joining the centres of mass of these two clusters defines the symmetry axis of the nucleus. Alternatively, we can say that only if this kind

† As is well known in δ-interactions the especially important Majorana force (position exchange) is equal to the Wigner force (no exchange).

of separation is possible, can we define a symmetry axis of a nucleus. Usually these two ultimate clusters are different from each other and also internally highly excited in general, but they must be spherically symmetrical around their own centres of mass; i.e. the internal angular moment of the clusters are zero. As has already been pointed out in I this separation in clusters is not always unique, because after antisymmetrization many different cluster structures may coincide.

If we again use our parameter representation, we can describe the wave function ψ of such an even nucleus as a superposition of internal wave functions: i.e. we obtain analogously to (6a)

$$\psi = \int \psi_{\text{int}}(\mathbf{r}_i) Y_{l,m}(\vartheta', \varphi') \mathrm{d}\cos\vartheta' \mathrm{d}\varphi'; \tag{33a}$$

ϑ', φ' are parameter coordinates which define the direction of the symmetry axis, ψ_{int} is the completely antisymmetrized internal wave function of the nucleus, l and m refer to its angular momentum. With this wave function it follows from invariance considerations, as was pointed out on p. 456, that the expectation value of the energy must have the following general form:

$$N^2\langle E\rangle = \int Y_{l,m}(\vartheta'', \varphi'') f(\gamma) Y_{l,m}(\vartheta', \varphi') \mathrm{d}\Omega'' \mathrm{d}\Omega'; \tag{33b}$$

$f(\gamma)$ is the transition matrix element of the total energy $\langle \vartheta'', \varphi'' | E | \vartheta', \varphi' \rangle$ for the directions ϑ'', φ'' and ϑ', φ'. It depends only on the angle γ between the directions ϑ'', φ'' and ϑ', φ'. The factor N^2 has a completely analogous form:

$$N^2 = \int Y^*_{l,m}(\vartheta'', \varphi'') g(\gamma) Y_{l,m}(\vartheta', \varphi') \mathrm{d}\Omega'' \mathrm{d}\Omega'. \tag{33c}$$

Thus, the expectation value of the energy has the same general form as in our special example. For rotational states $f(\gamma)$ and $g(\gamma)$ are independent of l, m. See also the considerations in the next section.

As we go over from lighter to heavier nuclei the number of nucleons in the two clusters, whose centres of mass define the symmetry axis of the nucleus under consideration, increases. This means that the number of nucleons which contribute to large deformations of heavy nuclei is relatively large. Also the amplitude of the relative oscillation of the two clusters increases in general with increasing atomic weight †, because the Pauli principle requires more and more nucleons to be raised into higher shells with increasing atomic weight (see for this I). Both effects are responsible for the fact that for heavy

† Exceptions occur in the neighbourhood of closed shells.

†† The direct terms in $f(\gamma)$ and $g(\gamma)$ are again δ-functions of the angle γ.

††† Every integration over a nucleon co-ordinate gives rise, in these transition matrix elements, to a factor which has on the average its maximum for $\gamma = 0$. The larger the relative oscillation amplitude of the two clusters which define the symmetry axis, the faster this factor decreases with increasing γ. Therefore a large number of nucleons in both clusters and a large relative oscillation amplitude create a factor in the integration over all nucleon co-ordinates which approaches zero very fast with increasing γ. See on this point the considerations of Yoccoz and Peierls [3]).

nuclei with large deformations the transition matrix elements $f(\gamma)$ and $g(\gamma)$ †† approach zero very rapidly with increasing γ †††. For these nuclei, therefore, the same approximations as in our special example can be applied in the consideration of large relative oscillation amplitudes. This means that for heavy nuclei with large deformations we have practically always to expect a rotational spectrum of the second kind for the lowest energy levels. From parity considerations it follows that as with Be^8 only even angular momenta are possible.

For nuclei near to the closed shells and at the closed shells the internal wave functions become more and more spherically symmetrical; therefore, our considerations are not valid for these nuclei. Indeed, if we describe for instance nuclei with closed shells approximately by oscillator wave functions, the relative oscillations of the two clusters defining the symmetry axis degenerates to a zero point oscillation: the orientation of the line joining the centres of mass of the clusters is thus lost. Because of the Pauli principle this can only occur if the internal excitations of these clusters are very high. If we consider for instance the ground state of O^{16} as an oscillation of two Be^8 clusters against each other, the internal excitation of the relative oscillations of the two α-clusters in the Be^8 clusters is a 6s-oscillation. Nuclei with closed shells and neighbouring nuclei therefore have no low energy rotational spectra, unless rotational spectra of the first kind occur accidentally.

Now we shall discuss briefly the essential difference of our method from the method of Yoccoz and Peierls [3]). Yoccoz and Peierls construct their internal wave functions in such a way that a rotationally symmetrical potential is given, in which the nucleons are moving. From this it follows that the real symmetry axis of their internal wave functions oscillates around the symmetry axis of their given potential. This has the consequence that in practical cases it is very difficult to split off the centre of mass motion of the whole nucleus in a clear way. Only if the deformation of the potential and the number of the nucleons in the nucleus is large is this oscillation negligible to a first approximation. Therefore this method is not very suitable for constructing rotational wave functions for nuclei with small deformations and for light nuclei. In our method we do not have this trouble. Here we have fixed directly the symmetry axis of the internal wave functions, and therefore we can split off the centre of mass motion in a clear way: the whole uncertainty of the direction of the nuclear symmetry axis is described by the spherical harmonics $Y_{l,m}$. The above-mentioned oscillation of the internal symmetry axis in the method of Yoccoz and Peierls appears also in the fact that in this method, in contrast to ours, one already gets overlapping terms if the nucleons are considered as distinguishable and no exchange forces are present. It can be shown — for instance explicitly in our special example — that for large deformations and for many nucleons in the

nuclei the results of both methods approximate each other. For instance the deviations in the rotational energies are of the order of magnitude $E_{\mathrm{rot}}(\Delta R/R)^2$, as a rough estimation shows; ΔR is the amplitude of the zero point oscillation of the centre of mass of the whole nucleus in the given rotationally symmetrical potential, and R is the nuclear radius.

As already mentioned, for heavy nuclei with large deformations, the influence of the exchange terms on the rotational energies is no longer negligible because of the strong interpenetration of the clusters. It seems possible to obtain agreement with the experimental rotational energies of both kinds to about 20 % with the Serber "ansatz" for the nuclear forces, which is given in I, if we assign the radii of the nuclei and their deformations phenomenologically. The calculations for Be^8 in I and the results of Yoccoz and Peierls [3]) support this: Yoccoz and Peierls obtain their best values for the rotational energies with a Serber "ansatz" for the nuclear forces.

We should mention that in large strongly deformed nuclei the kinetic and the potential rotational energy differences are increased by the antisymmetrization. In particular, the important position exchange forces which are present besides the Wigner forces enlarge the distances of the rotational levels, since at larger relative velocities these exchange forces become less and less attractive, as is well known (see for instance ref.[6])). Both effects can be demonstrated explicitly in our special example †: essentially we have to compare C_1, C_2 and N^2 with each other.

4. Electrical Quadrupole Transitions

The two different kinds of rotational levels which we have discussed in the preceding sections are also different in the general form of their associated γ-transition probabilities.

Here we only assume about the nuclear wave functions that they are rotational wave functions for even nuclei. This means in particular that their internal wave functions ψ_{int} are independent of the angular momenta. If we consider a rotational state with the angular momentum $I = l+2$, $I_z = l+2$, then for electric quadrupole radiation only transitions to states with $I = l$, $I_z = l$ are possible. The quadrupole operator for these transitions has the following form (see for instance ref.[7]), p. 102):

$$Q_{\mathrm{e}}(2,2) = Q_{\mathrm{e}}(\mathbf{r}_{\mathrm{p}}) = \tfrac{1}{4}e\sqrt{\frac{15}{2\pi}}\sum_{\mathrm{p}}(x_{\mathrm{p}}+iy_{\mathrm{p}})^2; \tag{34}$$

$x_{\mathrm{p}}, y_{\mathrm{p}}, z_{\mathrm{p}}$ are the coordinates of the protons relative to the centre of mass of the nucleus, e is the charge of a proton.

† It can be shown generally that in this special example the contribution of the Wigner forces to the energy is exactly the same as the contribution of the Majorana forces, while the contribution of the Heisenberg and Bartlett forces vanishes for a Serber "ansatz".

If we use for the wave functions the parameter representation (6) generalized for any number of particles, we obtain the following transition matrix element for electric quadrupole radiation:

$$M_e(l+2, l+2 \to l, l) = \tfrac{1}{4}\sqrt{\frac{15}{2\pi}}\int\Big[\Big\{A\int\psi^*_{l+2,\,l+2}(\mathbf{r}_i, \mathbf{r}''_i)\mathrm{d}\mathbf{r}''_i\Big\}Q_e(\mathbf{r}_p) \cdot \Big\{A\int\psi_{l,l}(\mathbf{r}_i, \mathbf{r}'_i)\mathrm{d}\mathbf{r}'_i\Big\}\Big]\,\mathrm{d}\mathbf{r}_i\frac{1}{N_{l+2,\,l+2}}\frac{1}{N_{l,l}}$$
$$= \tfrac{1}{4}A!\sqrt{\frac{15}{2\pi}}\int\Big[\Big\{A\int\psi^*_{l+2,\,l+2}(\mathbf{r}_i, \mathbf{r}''_i)\mathrm{d}\mathbf{r}''_i\Big\}Q_e(\mathbf{r}_p) \cdot \{\psi_{l,l}(\mathbf{r}_i, \mathbf{r}'_i)\}\mathrm{d}\mathbf{r}'_i\Big]\,\mathrm{d}\mathbf{r}_i\frac{1}{N_{l+2,\,l+2}}\frac{1}{N_{l,l}}; \tag{35a}$$

$\mathbf{r}_i$ denotes the coordinates of the nucleons, and $\mathbf{r}'_i$, $\mathbf{r}''_i$ the parameter coordinates which we have introduced in (6); A is the atomic weight of the nucleus; $N_{l+2,\,l+2}$ and $N_{l,l}$ are the normalization factors of the states $I = l+2$, $I_z = l+2$ and $I = l$, $I_z = l$; $\mathrm{d}\mathbf{r}_i$, $\mathrm{d}\mathbf{r}'_i$, $\mathrm{d}\mathbf{r}''_i$ denote the integration over the coordinates and parameter coordinates of all nucleons.

If we integrate over the coordinates of all nucleons, we obtain

$$M_e(l+2, l+2 \to l, l) = \tfrac{1}{4}A!\sqrt{\frac{15}{2\pi}}\int\Big[\Big\{A\mathbf{r}'_i\int\psi^*_{l+2,\,l+2}(\mathbf{r}'_i, \mathbf{r}''_i)\mathrm{d}\mathbf{r}''_i\Big\} \cdot Q_e(\mathbf{r}'_p)\psi_{l,l}(\mathbf{r}'_i)\Big]\,\mathrm{d}\mathbf{r}'_i\frac{1}{N_{l+2,\,l+2}}\frac{1}{N_{l,l}}. \tag{35b}$$

If we introduce, by means of the Eulerian angles, internal coordinates of the nucleons, and choose as the internal z-axis the symmetry axis of the wave function $\psi_{l,l}(\mathbf{r}'_i)$, we obtain

$$\begin{aligned}Q_e(\mathbf{r}'_p) = \tfrac{1}{4}e\sqrt{\frac{15}{2\pi}}\mathrm{e}^{2i\Psi'}\Big[&\sum_p \bar{x}_p^2\{\cos^2\chi' - \sin^2\chi'\cos^2\vartheta' + 2i\sin\chi'\cos\chi'\cos\vartheta'\}\\ &+\sum_p \bar{y}_p^2\{\sin^2\chi' - \cos^2\chi'\cos^2\vartheta' - 2i\sin\chi'\cos\chi'\cos\vartheta'\}\\ &-\sum_p \bar{z}_p^2\sin^2\vartheta' - 2\sum_p \bar{x}_p\bar{y}_p\{\cos\chi'\sin\chi'(1+\cos^2\vartheta') + i\cos\vartheta'(2\sin^2\chi' - 1)\}\\ &+2\sum_p \bar{z}_p\bar{x}_p\sin\vartheta'\{\sin\chi'\cos\vartheta' - i\sin\vartheta'\}\\ &+2\sum_p \bar{y}_p\bar{z}_p\sin\vartheta'\{\cos\chi'\cos\vartheta' + i\sin\chi'\}\Big].\end{aligned} \tag{36}$$

For the meaning of the notations introduced in (36) see fig. 1.

Now for the integration in eq. (35b) over the internal coordinates, at fixed symmetry axes of the two internal wave functions, we can choose the angle χ' in such a way that the $\bar{x}$-axis is always perpendicular to these symmetry

axes †. Then the transition matrix elements of $\sum_p \bar{x}_p^2$, $\sum_p \bar{y}_p^2$, $\sum_p \bar{z}_p^2$, etc. which belong to these internal wave functions, are only functions of the angle γ between their two symmetry axes. Furthermore it can be seen that the transition matrix elements of $\sum_p \bar{x}_p \bar{y}_p$ and $\sum_p \bar{z}_p \bar{x}_p$ must vanish for symmetry reasons.

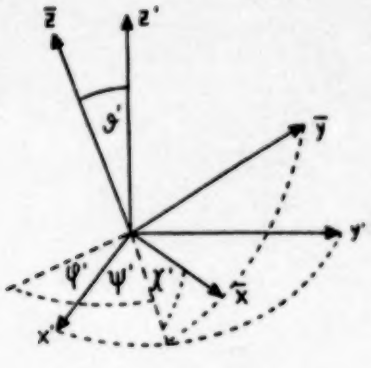

Fig. 1

If we consider at first the limiting case of heavy nuclei with large deformations, we can again apply the argument involving the overlapping of the internal wave functions. This implies that the above-mentioned transition matrix elements differ essentially from zero only if γ is very small. In this case the transition matrix elements of $\sum_p \bar{x}_p^2$ and $\sum_p \bar{y}_p^2$ are equal to each other in the first approximation, because the two symmetry axes nearly fall together. The transition matrix element of $\sum_p \bar{y}_p \bar{z}_p$ vanishes in this approximation ††. If we introduce $\psi' = \frac{1}{2}\pi + \varphi'$ (see fig. 1), we can write as before (see p. 462) for (35):

$$M_e(l+2, l+2 \to l, l) = e \int Y_{l+2,l+2}(\vartheta'', \varphi'') h(\gamma) Y_{l,l}(\vartheta', \varphi') \cdot Y_{2,2}(\vartheta', \varphi') \mathrm{d}\Omega'' \mathrm{d}\Omega' \frac{1}{N_{l+2,l+2}} \frac{1}{N_{l,l}}; \tag{37}$$

$h(\gamma)$ is the transition matrix element of the operator

$$\sum_p (\bar{x}_p^2 - \bar{z}_p^2) \approx \tfrac{1}{2} \sum (3\bar{z}_p^2 - \bar{r}_p^2) \tag{38}$$

corresponding to the two directions ϑ'', φ'' and ϑ', φ'. As on p. 458 we can represent $h(\gamma)$ as a sum of products of spherical harmonics, which depend on the angles ϑ'', φ'' and ϑ', φ'. If we do this, we obtain for (37):

† The angle χ' thus becomes a function of the polar coordinates ϑ'', φ'', ϑ', φ' of the two symmetry axes.

†† For this conclusion it is also important that $\Sigma \bar{x}_p^2$, $\Sigma \bar{y}_p^2$ and $\Sigma \bar{x}_p \bar{y}_p$ be sums of one-particle or two-particle operators, i.e. of operators which only depend on a small number of particle coordinates.

$$M_e(l+2, l+2 \to l, l) = g_{l+2} \frac{2\sqrt{\pi}}{\sqrt{2l+5}} \int Y^*_{l+2,\,l+2}(\vartheta', \varphi') Y_{l,l}(\vartheta', \varphi') \cdot Y_{2,2}(\vartheta', \varphi') \mathrm{d}\Omega' \frac{1}{N_{l+2,\,l+2}} \frac{1}{N_{l,l}} \tag{39}$$

with

$$g_{l+2} = 2\pi \int h(\gamma) Y^*_{l+2,0}(\cos\gamma) \mathrm{d}\cos\gamma.$$

For the integral over spherical harmonics we obtain, for instance by means of the tables of Condon and Shortley, ref. [8]), p. 77,

$$\int Y^*_{l+2,\,l+2}(\vartheta', \varphi') Y_{l,l}(\vartheta', \varphi') Y_{2,2}(\vartheta', \varphi') \mathrm{d}\Omega' = \tfrac{1}{2} \sqrt{\frac{15}{2\pi}} \sqrt{\frac{(l+1)(l+2)}{(2l+3)(2l+5)}}, \tag{40}$$

and therefore

$$M_e(l+2, l+2 \to l, l) = g_{l+2} \frac{2\sqrt{\pi}}{\sqrt{2l+5}} \frac{1}{N_{l+2,\,l+2}} \frac{1}{N_{l,l}} \frac{1}{2} \sqrt{\frac{15}{4\pi}} \sqrt{\frac{(l+1)(l+2)}{(2l+3)(2l+5)}}. \tag{41}$$

For the approximate calculation of g_{l+2}, analogous to (27), we expand $Y_{l+2,0}(\cos\gamma) \sin\gamma$ in a power series in γ. From (27) we can see that we can already neglect the first angular momentum dependent term in the integration over γ (if l is not too large) because of the factor γ^3 in this term †. It can be shown in a completely analogous manner that in the normalization factors $N_{l+2,\,l+2}$ and $N_{l,\,l}$ the angular momentum dependent parts can also be neglected in first approximation. Thus, we can assume $N_{l+2,\,l+2} \approx N_{l,\,l} \approx N_{0,0} = N$, and obtain for large nuclear deformations

$$M_e(l+2, l+2 \to l, l) \approx C \sqrt{\frac{(l+1)(l+2)}{(2l+3)(2l+5)}}$$

and

$$\frac{|M_e(l+2, l+2 \to l, l)|^2}{|M_e(l, l \to l-2, l-2)|^2} \approx \frac{(l+1)(l+2)(2l-1)(2l+1)}{(2l+3)(2l+5)l(l-1)}. \tag{42}$$

This is exactly the angular momentum dependence which one expects from the collective model. However, one obtains the value of C^2 given by the collective model,

$$C^2 = \frac{15}{32\pi} e^2 Q_0^2 \tag{43}$$

with

$$Q_0 = \int \psi^*_{\mathrm{int}}(\bar{\mathbf{r}}_i) \sum_{\mathrm{p}} (3\bar{z}_{\mathrm{p}}^2 - \bar{r}_{\mathrm{p}}^2) \psi_{\mathrm{int}}(\bar{\mathbf{r}}_i) \mathrm{d}\bar{\mathbf{r}}_i,$$

only if the nucleons are treated as distinguishable, in other words, if the exchange terms vanish and only the direct terms remain. As can easily be seen,

† In contrast to this the rotational energy differences depend directly on this first angular momentum dependent term.

the indistinguishability of the nucleons reduces the value C^2 of eq. (43).

If we now consider light nuclei, i.e. nuclei with a relatively small number of nucleons, the approximations which we have made until now are no longer allowed, because the transition matrix elements of $\sum \bar{x}_p^2$, $\sum \bar{y}_p^2$ and $\sum \bar{y}_p \bar{z}_p$ no longer vanish even for large values of γ. Therefore we can no longer put the transition matrix elements of $\sum \bar{x}_p^2$ approximately equal to those of $\sum \bar{y}_p^2$, nor can we any longer put the transition matrix elements of $\sum \bar{y}_p \bar{z}_p$ approximately equal to zero. Furthermore, for the same reason, the expansion of the transition matrix elements in spherical harmonics $Y_{l,0}(\cos\gamma)$ cannot be broken off after the angular momentum independent term. Therefore, it is no longer possible to give a general formula for the ratios of the reduced electrical quadrupole transition probabilities. It is necessary to calculate these ratios for every light nucleus separately.

For instance for Be^8 one obtains, at the lowest excitation levels on the basis of the oscillator model, for $|M_e(4 \to 2)|^2/|M_e(2 \to 0)|^2$ the strongly reduced value $\frac{45}{49}$ instead of the value $\frac{70}{49}$ of the collective model. The reduced value $\frac{45}{49}$ is explainable qualitatively by the fact that the ground state of Be^8 and the two lowest excited levels have the same kinetic energy. Thus, the mean velocity of the nucleons, which in the classical picture is responsible for the electro-magnetic radiation, does not become larger in the transition from the $l = 0$ to the $l = 2$ level than from the $l = 2$ to the $l = 4$ level; whereas, in the collective model, the increase of the mean velocity of the nucleons is much larger in the transition from the $l = 2$ to the $l = 4$ level than in the transition from the $l = 0$ to the $l = 2$ level. Therefore the reduced transition probability becomes relatively large for the $l = 4 \to l = 2$ transition on the basis of the normal collective model.

It is to be expected that in the rotational levels of light nuclei the relative increase of the kinetic energies with angular momentum is always strongly suppressed † compared with the kinetic energies of the collective model. Therefore, it is to be expected that the ratios of the associated reduced transition probabilities are always diminished compared with the same ratios on the collective model. Accordingly, the measurement of the ratios of these reduced transition probabilities can give information as to which of the two kinds of rotational spectra is present in a specific nucleus. For instance, it would be very interesting to know what kind of rotational levels those of Mg^{24} might be [9]). Their excitation energies are around 10 MeV.

We should mention again that the deviations from the rotational spectra for the wave functions which we have discussed here only arise from the indistinguishability of the nucleons and the exchange forces.

† If in the light nuclei rotational spectra appear, the rotational levels of a band very probably have always the same kinetic energy on the basis of the oscillator model. See also the examples in section 6.

5. Questions of Consistency

During our whole discussion we have used the following procedure: first we have constructed wave functions which have the symmetry properties we have to expect for wave functions describing rotational states. Afterwards we have investigated which conditions must be fulfilled for the associated energy spectra to be rotational spectra. Until now we have not discussed the question of how these wave functions can be derived from the interaction forces between the nucleons †.

Within the scope of the present investigation, such a derivation cannot be carried out completely, because we have always assumed, at least implicitly, that the interaction forces between the nucleons have no hard core. Such forces cannot give saturation ††; therefore, the saturation of the nuclear forces has to be taken into account roughly by fixing the radii of the nuclei phenomenologically (see I). The deformation of the nuclei in the ground state can be derived approximately, on the basis of the collective model [2]), from the interaction of the external nucleons with the closed shells, or by calculation of the energetically favoured cluster structure, as was done for Be^8 †††. At the moment, for heavy nuclei the second kind of calculations is practically impossible because of the large number of nucleons. For these calculations suitable approximation methods must first be developed. A third possibility is, as described in the papers of Yoccoz and Peierls, to express the deformation of the nuclei in the ground state phenomenologically. For deformed nuclei it is practically always to be expected that the lowest excited states can be described approximately by rotational wave functions, because for a given angular momentum these excitations are energetically most favoured. As our considerations have shown, for light nuclei these states very often do not have a rotational energy spectrum; we shall give some examples later. All other kinds of excitations are in most cases energetically much less favoured, because they are at least connected with energetically unfavoured surface changes of the nuclei (for instance oscillation excitations of higher order)*. Therefore, the admixture of these other kinds of excited states to the lowest nuclear states is generally relative-

† In most papers about the rotational states of the nuclei this problem is especially considered [10]).

†† Unless we assume forces which are incompatible with the low-energy two-nucleon scattering data.

††† I.e. one calculates the energetically favoured angular correlations of the nucleons in the nuclei (see I).

* *Note added in proof:* Another way to recognize the energetically favoured nature of the rotational states is the following. In going from one rotational state to the next the energetically favoured correlations between the nucleons are changed as little as possible. In all other excitation modes the change of the correlations between the nucleons is much greater. Hence, because of the short range of the nuclear forces, the energy gap in the latter cases is much larger.

ly small, as long as the angular momenta of the rotational states are not too large (for this also see Bohr and Mottelson [2]) †).

6. Further Examples of Nuclear States with Rotational Wave Functions

In this section we shall discuss briefly low energy states of some light nuclei, which can be described approximately by rotational wave functions.

It is to be expected that the low energy states of positive parity of F^{19} can be described approximately by unexcited O^{16} and H^3 clusters, because both clusters have relatively high first excited states as free particles. If we neglect the spin of the H^3 at first, we can expect, analogously to the considerations for Be^8 (see I), the level scheme represented in fig. 2.

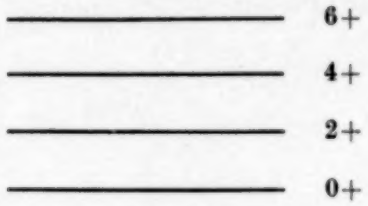

Fig. 2

On the basis of the oscillator model the 6s, 6d, 6g, 6f relative oscillations of the two clusters †† are also equivalent to rotational excitations, after antisymmetrization of the wave functions, if we assume for the radial part of the rotational wave functions

$$f(R) = R^6 e^{-(24/19)\alpha R^2} \tag{44}$$

where

$$\mathbf{R} = \frac{1}{3}\sum_{i=1}^{3} \mathbf{r}_i - \frac{1}{16}\sum_{i=4}^{19} \mathbf{r}_i .$$

If we take into account the spin of H^3 and the spin orbit coupling of the H^3 cluster with the O^{11} cluster, we get the splitting of the energy levels represented in fig. 3.

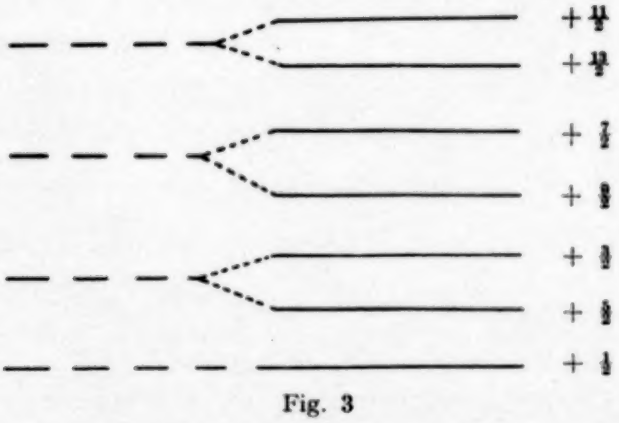

Fig. 3

† These deviations from the rotational spectra increasing with increasing l are generally much more important than the deviations mentioned on p. 460.

†† Relative oscillations of lower order are forbidden by the Pauli principle.

As far as has been experimentally ascertained (at least up to the $+\frac{3}{2}$ level), the lowest excitation levels of F^{19} with positive parity are in agreement with this scheme †. The fact that assuming this cluster structure, the calculated magnetic moments are in agreement with the experimental values strongly supports the cluster interpretation of the $+\frac{1}{2}$ and $+\frac{5}{2}$ levels. It must be mentioned that the complete antisymmetrization of the wave functions is very important for the calculation of the magnetic moments; without it the results are very often completely wrong. The level scheme of fig. 3 was investigated on the basis of the shell model by Elliot and Flowers [11]) and on the collective model by Paul [12]).

Very probably the very low lying $-\frac{1}{2}$ level of F^{19} can be described approximately by a N^{15} and α-particle clustering, because this cluster structure is very favoured energetically ††. If this interpretation is correct, it should give an excitation spectrum similar to that of fig. 3, with negative parity. Experimentally nothing can be said about it at the moment. Another test for this proposed structure of the $-\frac{1}{2}$ level would be the measurement of the magnetic moment of this level.

In a similar way the low-lying levels of many nuclei can be described. For instance, the lowest energy levels of Li^7 can be described by an α-particle and H^3 clustering approximately †††, *. The experimentally measured magnetic moment of the ground state of Li^7 is in good agreement with this assumption.

It is very probable that all low energy levels of nuclei possess strongly collective features (rotational or oscillation excitations). The regular structure of the low energy levels of the even nuclei strongly support this (the energy sequence of the levels is usually $+0$, $+2$, $+4$ . . .). In the odd-mass nuclei an unpaired nucleon is coupled to an even cluster. Because of the different coupling possibilities of the unpaired nucleon the collective structure of the low energy nuclear states very often does not appear in the energy spectra of the odd-mass nuclei. The Pauli principle together with the indistinguishability of the nucleons reduces the number of these coupling possibilities very strongly. An example of this is the low energy spectrum of Be^9 compared with that of Be^8. De Shalit [15]) in particular has investigated the energy spectra of odd-mass nuclei and compared them with those of the neighbouring even nuclei under the assumption of a strong j-j-coupling (see also Bohr and Mottelson [16])).

† We want to mention that the spin orbit coupling in F^{19} is not yet strong enough to interchange the sequence of the $+\frac{1}{2}$ and $+\frac{5}{2}$ levels, as one would expect on the basis of the simple shell model.

†† At present we are investigating this level especially.

††† This proposal for the ground state of Li^7 was made a long time ago by Inglis [13]).

* As is pointed out in the preceding considerations a clearly marked rotational energy spectrum is tied with the rotational wave functions of the light nuclei only in special cases. A review of rotational spectra in light nuclei can be found in a paper by Rakavy [14]).

7. Concluding Remarks

In the foregoing paper we pointed out that two different kinds of rotational states exist. The first kind is typical of many low energy levels of light nuclei, but only exceptionally shows a clearly marked rotational structure. The second kind of rotational level appears rather regularly in strongly deformed heavy nuclei and always shows a rotational structure of their excitation energies. If the rotational levels of the first kind have a rotational energy spectrum, then this spectrum always breaks off abruptly. For the rotational spectra of the second kind this is not the case. Here the gradual destruction of the rotational spectra with increasing angular momentum is caused essentially by the fact that excited states of another kind (for instance oscillation excitation) have excitation energies of the same order of magnitude as the high angular momentum rotational states. This creates a mixture of states having the same parity and the same angular momentum (centrifugal effect).

A further difference between the two kinds of rotational levels is that the ratios of the reduced transition probabilities which belong to the rotational levels of heavy even nuclei can be described by means of the collective model. The corresponding ratios are strongly reduced for the rotational levels of the first kind †. This difference should be particularly useful for distinguishing these two kinds of rotational levels.

By means of the method which we have applied here for the description of the collective nuclear states, it has also been possible to see the partition of the excitation energies of these states into kinetic and potential energy. We saw that the total potential energy of the rotational energies comes from the indistinguishability of the nucleons and from the exchange forces (antisymmetrization of the wave functions). The indistinguishability of the nucleons implies that the angles which define the direction of the nuclear symmetry axis can no longer be split off as collective co-ordinates of the nucleons by means of a product "ansatz" for the nuclear wave function in a simple way. In other words we cannot make, for the nuclear wave function, the "ansatz" $\phi(\bar{\mathbf{r}}_i, \vartheta, \varphi)\chi(\vartheta, \varphi)$ in which $\phi(\bar{\mathbf{r}}_i, \vartheta, \varphi)$ remains the same function for all states of a rotational band (assumption of the cranking model). If we do not take into account the exchange forces and the indistinguishability of the nucleons we always obtain essentially the moment of inertia of a rigid body. This moment of inertia is not always uniquely defined, because to a given nuclear state, there usually corresponds more than one cluster structure, as has been mentioned on p. 462.

† We want to mention that the rotational levels which we have considered here are also closely related to the rotational levels, which Elliott, Bohr and Mottelson in particular, have investigated [17]).

It seems that one can obtain the rotational energies of the nuclei to about 20 % if one makes for the nuclear forces a Serber "ansatz" which describes the low energy two-body scattering data to a good approximation, and if one specifies either the nuclear deformations or the cluster structure of the nuclei. In light nuclei it seems possible, with such two-body forces, to calculate rather well the energy separations of levels which have the same cluster structure (without internal cluster excitation) and relative oscillations of the same order between the clusters †. It is necessary to add a spin-orbit coupling term to these forces. On the contrary, the energy distances of levels with different cluster structure and different order of their relative oscillations are badly described ††. The shell model calculations of Inglis [18]) Kurath [19]), and Elliot and Flowers [11]), together with our calculations in I, indicate these facts.

As already mentioned, it is possible that the method which we have applied here can be generalized without violation of the conservation laws of momentum and angular momentum. For instance, we can take into account polarization effects (especially in F^{19}) choosing the width parameters of the clusters and of the relative oscillations different from each other. We can also extend this method to a general investigation of the rotational spectra of odd-mass nuclei and other kinds of collective excitation.

Another possible application of this method is the derivation of the Schrödinger equation for the scattering of two α-particles, if the internal excitation of the α-clusters is neglected †††. One obtains for this Schrödinger equation an integro-differential-equation which contains also the velocity dependence of the interaction forces between the two α-clusters *.

Finally we would like to mention some points which we have not made clear enough in I. The centre of mass kinetic energy of the clusters in a nucleus is very often much larger than the internal kinetic energy of a nucleon in a cluster. For instance, in I we have seen that the centre of mass kinetic energy of the two α-clusters in the Be^8 ground state is about 38 MeV, whereas the zero-point kinetic energy of a nucleon in an α-particle is about 20—30 MeV. From this it can be seen that the usual adiabatic condition applied to the collective motions of the nuclei is too restricted. As is well known this condition means that one can expect well marked collective

† For instance for Be^8 the energy distances of the 4s, 4d, 4g levels, and for F^{19} the energy distances of the 6s, 6d, 6g levels.

†† Here it is a bad approximation to take for different states the same width parameter of the oscillator potential. For instance, we cannot expect to be able to calculate the energy difference of the $+\frac{1}{2}$ and $-\frac{1}{2}$ level of F^{19} to a good approximation.

††† For scattering energies up to 15 MeV this neglect is justified to a good approximation, as the calculations of I show.

* We shall discuss in a forthcoming paper this Schrödinger equation in detail.

motions only if the internal velocities of the nucleons in the nuclei are much larger than the mean velocities of the collective motions.

As already mentioned in I the differences between the various cluster structures of a nucleus are very much reduced by the antisymmetrization of the wave functions. After this antisymmetrization only small residual effects remain of the conglomeration within the different inter-penetrating particle groups†. However, these residual effects can be responsible for favouring certain nuclear reactions. For instance, it is to be expected that at sufficiently high excitation of the nuclei more α-particles are emitted than would be predicted by the usual nuclear evaporation theory. Other examples are F^{19} and Na^{23}: here it is to be expected that at sufficiently high excitation energy more tritons are emitted than follows from the evaporation theory; this is connected with the important fact that we obtain the free particles belonging to the clusters in the nucleus by an adiabatic change of the relative motions of the clusters and an adiabatic change of their radii. In most cases the differences between the different cluster structures of a nucleus are the larger the more excited the nucleus is.

In conclusion we want to stress again that to make our considerations more complete, it is necessary to introduce the saturation of the nuclear forces in a consistent way, but to find a practical way for doing this is very difficult because of the hard core of the nuclear forces. However, it is perhaps possible to justify approximately the method of splitting off the hard core effects for a certain group of problems, as we have done in this paper and in I, by using the Brueckner theory in the formulation of Weisskopf and others [20])

We would like to thank the members of the CERN Theoretical Division for their helpful discussions and remarks. We thank Dr. P. Hillman for his very kind help in correcting our English.

† It follows, for instance, that the magnetic moment of a nucleus is usually completely different from the magnetic moment which we would obtain by simple vector addition of the magnetic moments of the clusters of the nucleus.

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STRANGE PARTICLE DECAYS AND THE UNIVERSAL *V—A* FOUR-FERMION INTERACTION

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Abstract: The branching ratios of various decay modes of K-meson are discussed on the basis of a universal vector and axial-vector four-fermion interaction. The simplest and most straightforward interpretation of the results leads to the following tentative conclusion:

The (squared) coupling constant of the strangeness non-conserving baryon-lepton interactions, such as the $\Lambda^0 \to p+e^-+\bar{\nu}$ or $\Sigma^- \to n+e^-+\bar{\nu}$, is smaller than the usual Fermi coupling constant $(1.4\times 10^{-49}\,\mathrm{erg}\cdot\mathrm{cm}^3)^2$ by, at least, one order of magnitude. This is rather consistent with the recent observations on the hyperon decays into leptons.

Alternative possibilities which may enable us to avoid this conclusion are also discussed. An example is to expect strong final state interactions in the $K^0 \to \pi^+ + \pi^-$ decay.

Also phenomenological estimates of the decay rates of $\Lambda^0 \to p+e^-(\mu^-)+\bar{\nu}$ from the experimentally known rates of $K \to \mu+\nu$ and $K \to \pi+e(\mu)+\nu$ decays are made. It is concluded that the computed lower limits of the rates are not in contradiction with the present experimental information.

1. Introduction

Recent experimental developments †† give a new light on the so called universal Fermi interaction. The β-decay [2]) and μ-decay phenomena are consistent with the form of direct four fermion interactions with the V—A combination [3,4,5]). The very close agreement (within a few percent) between the coupling constants for μ-meson decay and O^{14} decay implies that the renormalization effects are very small for the vector part of the interactions. The axial-vector part also suffers no significant renormalization effect [6]) (within 20 percent) for low energy β-transitions. The interaction $\mu^- + p \to n + \nu$ may as well be included in this scheme. This seems to be welcome also by the fact that the qualitative account of the mysterious behaviour of the pion decays is more or less satisfactory in this theory. We may write down, in an obvious notation, the V—A form of these interactions for *bare* fermions in the following way:

μ^--decay: $[\bar{\nu}\gamma_\alpha(1+\gamma_5)\mu][\bar{e}\gamma_\alpha(1+\gamma_5)\nu]+\mathrm{h.c.}$

β-decay: $[\bar{p}\gamma_\alpha(1+\gamma_5)n][\bar{e}\gamma_\alpha(1+\gamma_5)\nu]+\mathrm{h.c.}$

muon-capture: $[\bar{p}\gamma_\alpha(1+\gamma_5)n][\bar{\mu}\gamma_\alpha(1+\gamma_5)\nu]+\mathrm{h.c.}$

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†† For a recent compilation see, for instance, ref. [1]).

These interactions may be considered as the couplings between various *charged* currents:

$$j_\alpha(\mathrm{e}, \nu) = \bar{\mathrm{e}}\gamma_\alpha(1+\gamma_5)\nu \tag{1}$$

$$j_\alpha(\mu, \nu) = \bar{\mu}\gamma_\alpha(1+\gamma_5)\nu \tag{2}$$

$$J_\alpha^{(0)}(\mathrm{n}, \mathrm{p}) = \bar{\mathrm{n}}\gamma_\alpha(1+\gamma_5)\mathrm{p} \tag{3}$$

and the hermitian conjugate currents. These currents consist of vector and axial-vector parts and we may call them chiral currents [3]). We may even take such a viewpoint that the Fermi interaction can be considered as the *self-coupling* of a chiral current $\mathcal{J}_\alpha$ with a coupling constant $f/\sqrt{2}$ ($f \approx 1.4 \times 10^{-49}$ erg·cm^3):

$$f/\sqrt{2} \cdot \mathcal{J}_\alpha \mathcal{J}_\alpha^* + \mathrm{h.c.} \tag{4}$$

where the current $\mathcal{J}_\alpha$ itself is constructed additively from baryon and lepton parts such as given by (1)—(3). In addition to the current (3) there might exist other baryon currents which conserve the strangeness quantum number ($\Delta S = 0$):

$$\begin{aligned} J_\alpha^{(0)}(\Lambda^0 \text{ or } \Sigma^0, \Sigma^+) &= \bar{\Lambda}^0(\text{or } \bar{\Sigma}^0)\gamma_\alpha(1+\gamma_5)\Sigma^+ \\ J_\alpha^{(0)}(\Sigma^-, \Lambda^0 \text{ or } \Sigma^0) &= \bar{\Sigma}^-\gamma_\alpha(1+\gamma_5)\Lambda^0(\text{or } \Sigma^0) \\ J_\alpha^{(0)}(\Xi^-, \Xi^0) &= \bar{\Xi}^-\gamma_\alpha(1+\gamma_5)\Xi^0 \end{aligned} \tag{3'}$$

and the hermitian conjugate currents.

The striking fact that all the weak reactions, including those involving strange particles, seem to be characterized by coupling strengths of nearly the same order of magnitude may suggest that the notion of the universal V—A Fermi interactions can be extended to include the strange particles. In fact, we may add to the currents mentioned above (which preserve the strangeness quantum number) the currents which satisfy $|\Delta S| = 1$ (strangeness changing by one) in order to account for the various strange particle decays characterized by a rule $|\Delta S| = 1$:

$$\begin{aligned} J_\alpha^{(1)}(\Lambda^0 \text{ or } \Sigma^0, \mathrm{p}) &= \bar{\Lambda}^0(\text{or } \bar{\Sigma}^0)\ \gamma_\alpha(1+\gamma_5)\mathrm{p} \\ J_\alpha^{(1)}(\Sigma^-, \mathrm{n}) &= \bar{\Sigma}^-\gamma_\alpha(1+\gamma_5)\mathrm{n} \\ J_\alpha^{(1)}(\Xi^-, \Lambda^0 \text{ or } \Sigma^0) &= \bar{\Xi}^-\gamma_\alpha(1+\gamma_5)\Lambda^0(\text{or } \Sigma^0) \\ J_\alpha^{(1)}(\Xi^0, \Sigma^+) &= \bar{\Xi}^0\gamma_\alpha(1+\gamma_5)\Sigma^+, \end{aligned} \tag{5}$$

$$\begin{aligned} J_\alpha^{(1)}(\mathrm{n}, \Sigma^+) &= \bar{\mathrm{n}}\gamma_\alpha(1+\gamma_5)\Sigma^+ \\ J_\alpha^{(1)}(\Sigma^-, \Xi^0) &= \bar{\Sigma}^-\gamma_\alpha(1+\gamma_5)\Xi^0, \end{aligned} \tag{5'}$$

and the hermitian conjugate currents.

Then we may think that the chiral current $\mathcal{J}_\alpha$ in (4) may consist of the lepton current j_α, the strangeness conserving baryon current $\mathcal{J}_\alpha^{(0)}$ and the strangeness non-conserving baryon current $\mathcal{J}_\alpha^{(1)}$:

$$\mathcal{J}_\alpha = j_\alpha + \mathcal{J}_\alpha^{(0)} + \mathcal{J}_\alpha^{(1)}. \tag{6}$$

The currents (5) and (5′) are characterized by the property $\Delta S/\Delta Q = 1$ and $\Delta S/\Delta Q = -1$ respectively, where Q is the charge of the baryon system. If the property $\Delta S/\Delta Q = 1$ is favoured by nature we would not observe such decays as $\Sigma^+ \to n+e^++\nu$, $K^+ \to \pi^++\pi^++e^-(\mu^-)+\bar{\nu}$ etc.[4,7]). This is indeed the case with, for instance, Sakata's composite particle model of baryons [8]) or with an assumption that these strangeness non-conserving currents $\mathcal{J}_\alpha^{(1)}$ transform like a spinor in isobaric spin space [9]) (i.e. $|\Delta I| = \frac{1}{2}$). There are several direct consequences which may be regarded as supporting evidence on the above straightforward extension of the form (4) with (6) of the universal V−A Fermi coupling to strange particle interactions.

(a) As is the case with the pion decay, the effective form of the interactions, $K \to \mu+\nu$ and $K \to e+\nu$, presume a derivative type Yukawa interaction and so the frequency ratio of the $K \to e+\nu$ to $K \to \mu+\nu$ decay is about 2.5×10^{-5}, which seems to be favoured by present experiments. As will be shown in § 2 the comparable frequency of K_{e3} and $K_{\mu3}$ decays can also be easily explained.

(b) The angular distribution of the pion produced in the polarized hyperon decay is measured. The angular distribution has the form $1+p\alpha\cos\theta$. Here θ is the angle between the pion momentum and the polarization axis, p denotes the degree of polarization of the hyperon.

The asymmetry coefficient was found to be as large as $|p\alpha| \gtrsim 0.5$ in the case of $\Lambda^0 \to p+\pi^-$ decay [10]). For explaining this fact the dominant part of effective interaction hamiltonian should be a derivative type Yukawa interaction

$$\bar{p}\gamma_\alpha(1+\gamma_5)\Lambda\partial_\alpha\varphi_\pi+\text{h.c.} \tag{7}$$

if we assume that the parity violation occurs through its extremum form $1\pm\gamma_5$ as is the case with other weak interactions (1), (2) and (3). The Yukawa type direct interaction gives too small a coefficient even if we take into account the final state interactions [5,11,12]) †.

It has been pointed out [13]) that under certain approximations the V−A Fermi interaction between four bare baryons can easily lead to an effective hamiltonian of the form (7). In fact, we may pick up the following four-baryon couplings from the starting interaction (4) with (6) which are directly responsible for the $\Lambda^0 \to p+\pi^-$ decay:

$$\begin{aligned}&\frac{f}{\sqrt{2}}J_\alpha^{*(1)}(\Lambda^0, p)\mathcal{J}_\alpha^{(0)}\\ &\equiv\frac{f}{\sqrt{2}}J_\alpha^{*(1)}(\Lambda^0, p)\{J_\alpha^{(0)}(n, p)+J_\alpha^{(0)}(\Lambda^0, \Sigma^+)+J_\alpha^{(0)}(\Sigma^0, \Sigma^+)+\ldots\}.\end{aligned} \tag{8}$$

† Not much asymmetry has been observed so far in the Σ-decay, although the statistics are still poor. It is not clear whether this is a real difference or is due to the possibility that the polarization of Σ produced in $\pi+p \to \Sigma+K$ is not large enough to give us an observable asymmetry.

Here $\mathcal{J}_\alpha^{(0)}$ denotes the sum of all existing strangeness conserving currents ($\Delta S = 0$). Now, we shall investigate such types of matrix elements in which the current $\mathcal{J}_\alpha^{(0)}$ transforms into a π^--meson which are shown in fig. 1. Then the matrix elements of this type can be expressed in the form (7),

$$\mathfrak{g}\bar{p}\gamma_\alpha(1+\gamma_5)\Lambda\partial_\alpha\varphi_\pi+\text{h.c.} \tag{9}$$

where $\mathfrak{g}$ denotes the contribution of black circle in fig. 1 and is a scalar constant. This form of effective hamiltonian is, at least, valid up to higher order corrections as long as strong interactions acting on the Λ^0 and proton in the f-vertex are switched off [13]). Thus if the renormalization effects due to these neglected corrections are not significant, or if they do not change appreciably the covariant structure of (9), the V—A four-bare-fermion interaction seems to be able to explain the above mentioned experiments.

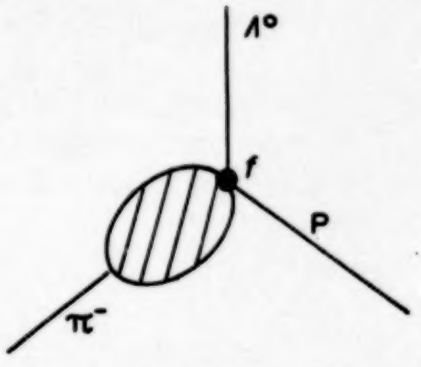

Fig. 1

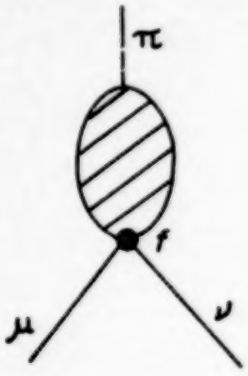

Fig. 2

(c) Further it has also once been noticed [14]) that all the coupling constants of the weak *pion-two-fermion* interactions come very close to each other when we assume the interactions of vector or axial vector forms. If we assume the form

$$g\bar{\psi}\gamma_\alpha(1+\gamma_5)\psi'\partial_\alpha\varphi_\pi+\text{h.c.} \tag{10}$$

the coupling constants g of various weak *pion-two-fermion* interactions take the values listed in table 1.

TABLE 1

Interaction	Experimental partial life-time (sec)	Coupling constant $(g^2/4\pi)m_\pi^2$
$\Lambda^0 \to p+\pi^-$ ($Q = 37$ MeV)	4.3×10^{-10}	4.3×10^{-15}
$\Sigma \to n+\pi$ ($Q = 110$ MeV)	1.45×10^{-10}	2.0×10^{-15}
$\Xi \to \Lambda+\pi$ ($Q = 65$ MeV)	4.6×10^{-10}	1.8×10^{-15} (assumed)
$\pi \to \mu+\nu$	2.5×10^{-8}	1.8×10^{-15}

The agreement between the various (squared) coupling constants is excellent within a factor ≈ 2. This is not so hard to understand on the basis of the

universal V—A four fermion interaction †. The interactions which are responsible for the $\pi \to \mu + \nu$ decay will be picked up from (4) as

$$\frac{f}{\sqrt{2}} j_\alpha^*(\mu, \nu) \mathscr{J}_\alpha^{(0)}. \tag{11}$$

Then, the matrix element of the $\pi \to \mu + \nu$ decay (see fig. 2) is given by

$$\mathfrak{g} \bar{\nu} \gamma_\alpha (1+\gamma_5) \mu \partial_\alpha \varphi_\pi + \text{h.c.} \tag{12}$$

The one-one correspondence between the black circles in fig. 1 and fig. 2 implies that the effective coupling constants $\mathfrak{g}$ which appear in (9) and (12) are the same.

This argument may indicate that the approximate equality of the coupling constants of *pion-two-fermion* interactions listed in table 1 could have their origin in the basic universal interaction of the form (4). In other words, if the form of weak universal interaction (4) assumed here is correct the renormalization effects which are neglected in the case of $\Lambda^0 \to \mathrm{p} + \pi^-$ should not be so large.

Thus far, the theory seems to be very successful. However, there seems to be some difficulties to be solved. One of them is the alternative modes of decays of hyperons into leptons. The V—A Fermi interactions predict the following partial decay rates for these processes [14,15]):

$$\Lambda^0 \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}: \quad 5.4 \times 10^7 \ \mathrm{sec}^{-1} \tag{13}$$

$$\Sigma^- \to \mathrm{n} + \mathrm{e}^- + \bar{\nu}: \quad 2.9 \times 10^8 \ \mathrm{sec}^{-1}, \tag{14}$$

where we have used the usual value of the coupling constant $f = 1.4 \times 10^{-49}$ erg · cm^3. Experimentally, there is as yet no clear-cut evidence for the hyperon decays into leptons. The upper limit †† on the β-decay of Λ^0 and Σ^- is experimentally, at least, one order of magnitude [16]) below the calculated rates (13) and (14). Under this circumstance there may be two distinct standpoints:

(I) The above experimental fact may not be a difficulty at all for having the same coupling constant for the bare-four-fermion interactions, since the renormalization effects due to strong interactions might account for the discrepancy mentioned above.

Under the inclusion of radiative corrections, the form of the relevant S-matrix may no longer be of pure V—A type but terms which have the form of pseudoscalar and scalar coupling and also terms which even involve derivatives of the baryon fields may be induced. It might happen that a fortuitous cancellation takes place among these terms so as to suppress the

† Essentially the same discussion has been developed with regard to the study of the universality of the *boson-two-fermion* interactions in ref.[13]). See also ref.[1]), p. 442.

†† *Note added in proof:* Two events of the type $\Lambda^0 \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}$ have recently been found by the Berkeley group. At present, the branching ratio of $\Lambda^0 \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}$ to $\Lambda^0 \to \mathrm{p} + \pi^-$ is reported to be $\approx 1/1500$, which is still smaller by about one order of magnitude than that given by (13). Report at the Conference on Weak Interactions, Gatlinburg, Tennessee, October, 1958.

decay rates of $\Lambda^0 \to p+e^-(\mu^-)+\bar{\nu}$ and $\Sigma^- \to n+e^-(\mu^-)+\bar{\nu}$ etc. It should, however, be kept in mind that the induced terms do, in general, depend on the available kinetic energy of the process and, therefore, it is rather hard to expect the occurrence of such cancellations both in the Λ-decay and Σ-decay and also both in the electron and μ-meson modes at the same time†.

(II) If it turns out that such a renormalization effect cannot be so important and that the accumulation of the experimental data does not change the present experimental results, the situation may not be so simple. If we want to keep the universal form of interactions (4) for these reactions (there is a supporting argument mentioned in (a) and an unambiguous experimental check could be done by using the $K^0 \to \pi^{\mp}+e^{\pm}+\nu$ decay [17]), we may be forced to conclude that the (squared) coupling constant of the strangeness non-conserving baryon-lepton part of the interaction (4), such as $\Lambda^0 \to p+e^-+\bar{\nu}$, $\Sigma^- \to n+e^-+\bar{\nu}$ etc., is smaller by at least a factor of about 10 than the square of the usual Fermi coupling constant.

This possibility of having a fine structure even in the weak coupling constants, though less attractive than the simpler standpoint (I), may exhibit some similarity to the situation concerning strong interactions. Gell-Mann [18]) and Schwinger [19]) proposed the theory of global symmetry. There, strong interactions involving K-mesons are assumed to have weaker coupling constants (by a factor ≈ 10) than universal pion-baryon interactions which presumably have the highest symmetry. Therefore, we should perhaps keep our mind open to the possibility of the standpoint (II).

The main purpose of this paper is to point out that the branching ratios of K^0-meson decay modes computed on the basis of V—A four-fermion interaction are rather consistent with the standpoint (II). That is, the observed branching ratios of the K-meson seem to be closely connected with the recently observed small decay rates of hyperon decays into leptons [16]).

2. Branching Ratios of K-meson Decays Based on the *V—A* Four-Fermion Interactions

First let us discuss the $K_{\mu 3}$ and K_{e3} modes. From (4) and (6) the needed interactions are

$$\mathcal{H}' = \frac{f}{\sqrt{2}} \mathcal{J}_\alpha^{(1)} j_\alpha^*(\mu \text{ or } e, \nu)+\text{h.c.} \tag{15}$$

where $\mathcal{J}_\alpha^{(1)}$ is the sum of existing strangeness non-conserving currents of the type (5) and (5′). Now we shall, at first, work under the assumption that the strangeness non-conserving currents $\mathcal{J}_\alpha^{(1)}$ (in order to forbid the reactions

† *Note added in proof:* S. Glashow and E. C. G. Sudarshan have found that the contribution of these extra momentum-dependent terms is not significant for any value of these additional parameters. Report at the Conference on Weak Interactions, Gatlinburg, Tennessee, October, 1958.

with $|\Delta S| = 2$) satisfy [4]) the condition

$$\Delta S/\Delta Q = 1. \tag{16}$$

Then currents of the type (5′) do not contribute to $\mathcal{J}_\alpha^{(1)}$. We shall, of course, neglect the electromagnetic corrections and, therefore, the lepton vertex is always localized (see fig. 3).

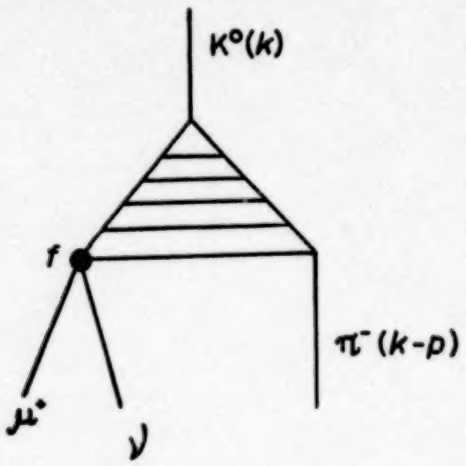

Fig. 3

The amplitude for the decay $K^0(k) \to \pi^-(k-p)+\mu^++\nu$ will be given by

$$\mathcal{M}^0(\mu) \equiv \mathcal{M}(K^0 \to \pi^- + \mu^+ + \nu) = \frac{f}{\sqrt{2}} \langle \pi^- | \mathcal{J}_\alpha^{(1)} | K^0 \rangle j_\alpha^*(\mu, \nu). \tag{17}$$

However, on account of condition (16) we have

$$\bar{\mathcal{M}}^0(\mu) \equiv \mathcal{M}(\overline{K}^0 \to \pi^- + \mu^+ + \nu) = 0. \tag{18}$$

The K^0-meson, however, is not a real particle but a mixture † of the particles K_1^0 and K_2^0 [20]),

$$K_1^0 = \frac{1}{\sqrt{2}}(K^0 + \overline{K}^0), \quad K_2^0 = \frac{1}{\sqrt{2}}(K^0 - \overline{K}^0). \tag{19}$$

Now we are interested in the transition probabilities of K_1^0 and K_2^0 into $\pi^- + \mu^+ + \nu$ plus $\pi^+ + \mu^- + \bar{\nu}$, denoted as $W(K_1^0 \to \pi^\mp + \mu^\pm + \nu)$ and $W(K_2^0 \to \pi^\mp + \mu^\pm + \nu)$ respectively. Using (16) (consequently (18)) and applying the TCP theorem to this case [21]) we have

$$\begin{aligned} W(K_1^0 \to \pi^\mp + \mu^\pm + \nu) &= W(K_2^0 \to \pi^\mp + \mu^\pm + \nu) \\ &= W(K^0 \to \pi^- + \mu^+ + \nu). \end{aligned} \tag{20}$$

The same formulae follow for the electron decay case also:

$$\begin{aligned} W(K_1^0 \to \pi^\mp + e^\pm + \nu) &= W(K_2^0 \to \pi^\mp + e^\pm + \nu) \\ &= W(K^0 \to \pi^- + e^+ + \nu). \end{aligned} \tag{21}$$

Direct measurements of the frequency ratio of the $K_1^0 \to \pi^\mp + \mu^\pm(e^\pm) + \nu$

† Here we have assumed time reversal invariance of the theory.

may be very difficult. However, using the relation (20) and (21) we can evaluate the probability $W(K_1^0 \to \pi^{\mp}+\mu^{\pm}(e^{\pm})+\nu)$ provided we get sufficiently accurate measurements of the K_2^0-decay modes. Recent measurements on the mean life-time of K_2^0-meson have given [22])

$$9.0^{+3.5}_{-2.5} \times 10^{-8} \text{ sec.} \tag{22}$$

It seems that the $K_2^0 \to \pi^{\mp}+\mu^{\pm}+\nu$, $\pi^{\mp}+e^{\pm}+\nu$ and $\pi^+ + \pi^- + \pi^0$ modes account for approximately 30 percent each of the total K_2^0-decay [1]). Thus we may have a crude estimate for (22):

$$W(K_2^0 \to \pi^{\mp}+\mu^{\pm}+\nu) \approx W(K_2^0 \to \pi^{\mp}+e^{\pm}+\nu) \approx 3.3^{-0.9}_{+1.3} \times 10^6 \text{ sec}^{-1}. \tag{23}$$

Then, from the present experimental values for the decay rate of K_1^0-meson[1]), 1.05×10^{10} sec^{-1} (the partial decay rate of $K_1^0 \to \pi^+ + \pi^-$ is 0.9×10^{10} sec^{-1}), we estimate that

$$\frac{W(K_1^0 \to \pi^{\mp}+\mu^{\pm}+\nu)}{W(K_1^0 \to \text{any mode})} \approx \frac{W(K_1^0 \to \pi^{\mp}+e^{\pm}+\nu)}{W(K_1^0 \to \text{any mode})} \approx 3.1 \times 10^{-4}. \tag{24}$$

It may be interesting to quote here the results of Okubo *et al.* [9]) who studied the consequence of the condition stronger, than (16) used in this section,

$$|\Delta I| = \tfrac{1}{2}, \tag{25}$$

imposed on the strangeness non-conserving current $\mathcal{J}_\alpha^{(1)}$. Then, the relation between the decay rates of $K^0_{\mu 3}$ and $K^+_{\mu 3}$ is obtained and (20) and (21) can be extended respectively to

$$\begin{aligned} &W(K_1^0 \to \pi^{\mp}+\mu^{\pm}+\nu) = W(K_2^0 \to \pi^{\mp}+\mu^{\pm}+\nu) \\ &= W(K^0 \to \pi^- + \mu^+ + \nu) = 2W(K^+ \to \pi^0 + \mu^+ + \nu) \end{aligned} \tag{26}$$

and

$$\begin{aligned} &W(K_1^0 \to \pi^{\mp}+e^{\pm}+\nu) = W(K_2^0 \to \pi^{\mp}+e^{\pm}+\nu) \\ &= W(K^0 \to \pi^- + e^+ + \nu) = 2W(K^+ \to \pi^0 + e^+ + \nu). \end{aligned} \tag{27}$$

Now, experimentally the decay rates of $K^+ \to \pi^0 + \mu^+ + \tilde{\nu}$ and $K^+ \to \pi^0 + e^+ + \tilde{\nu}$ are nearly equal [1]) ($\approx 3.3 \times 10^6$ sec^{-1}). Then from the relations (26) and (27) we must have

$$\begin{aligned} &W(K_2^0 \to \pi^{\mp}+\mu^{\pm}+\nu) \approx W(K_1^0 \to \pi^{\mp}+\mu^{\pm}+\nu) \\ &\approx W(K_2^0 \to \pi^{\mp}+e^{\pm}+\nu) \approx W(K_1^0 \to \pi^{\mp}+e^{\pm}+\nu) \approx 6.6 \times 10^6 \text{ sec}^{-1}. \end{aligned} \tag{28}$$

This is larger by about a factor two than the value given by (23). It would be too early to decide which of the conditions (16) or (25) is favoured by experiments. This will, however, be settled in the very near future. If we use the result (28) we have instead of (24),

$$\frac{W(K_1^0 \to \pi^{\mp}+\mu^{\pm}+\nu)}{W(K_1^0 \to \text{any mode})} \approx \frac{W(K_1^0 \to \pi^{\mp}+e^{\pm}+\nu)}{W(K_1^0 \to \text{any mode})} \approx 6.3 \times 10^{-4}. \tag{29}$$

It should also be noted that if the K-meson is pseudoscalar (scalar) only the

vector (axial-vector) part of the current $\mathcal{J}_\alpha^{(1)}$ contributes to these decays.

Now, we turn to evaluate the amplitude $\mathcal{M}^0$ given by (17). From general arguments of invariance for strong interactions we may write the $\mathcal{M}^0(\mu)$ in the form

$$\mathcal{M}^0(\mathrm{K}^0 \to \pi^- + \mu^+ + \nu) = \frac{f}{\sqrt{2}} \varphi_{\mathrm{K}}(k) \varphi_\pi^*(k-p)(k_\alpha M + p_\alpha N) j_\alpha^*(\mu, \nu) \quad (30)$$

where k_α and $(k-p)_\alpha$ denote the energy-momentum four-vector of the K-meson and pion respectively. M and N represent complicated contributions from the black triangular box of fig. 3 and are the scalar functions of four scalar $(kp) = -m_{\mathrm{K}}^2 + m_{\mathrm{K}} E_\pi$, i.e., of the pion energy E_π. By fixing this outgoing pion energy, M and N are, of course, constant. They can be taken as real numbers since time reversal invariance of strong interactions is assumed. As we shall see below it is important to realize that the quantities $|M|$ and $|N|$ are likely to be of the same order of magnitude. Using equations of motions of leptons we rewrite (30) into

$$\mathcal{M}^0(\mu) = \frac{f}{\sqrt{2}} \varphi_{\mathrm{K}}(k) \varphi_\pi^*(k-p) \{ M \bar{\nu}(\gamma k)(1+\gamma_5)\mu - N(i m_\mu)\bar{\nu}(1-\gamma_5)\mu \}. \quad (31)$$

Neglecting the electron mass, the amplitude for electron decay is simply given by

$$\mathcal{M}^0(\mathrm{e}) = \frac{f}{\sqrt{2}} \varphi_{\mathrm{K}}(k) \varphi_\pi^*(k-p) M \bar{\nu}(\gamma k)(1+\gamma_5)\mathrm{e}. \quad (32)$$

The transition rate of the $\mathrm{K}^0 \to \pi^- + \mu^+ + \nu$ for a fixed pion energy E_π is [8])

$$\begin{aligned} W(E_\pi)\mathrm{d}E_\pi = \left(\frac{1}{2\pi}\right)^3 \left|\frac{f}{\sqrt{2}}\right|^2 (m_\pi)^5 2 \frac{m_{\mathrm{K}}}{m_\pi} \Bigg[\left(\frac{m_{\mathrm{K}}}{m_\pi}\right)^2 M^2 \left\{ -I_1 + 2I_2 - \tfrac{2}{3}\frac{m_{\mathrm{K}}}{m_\pi} I_4 \right\} \\ + 2MN \frac{m_{\mathrm{K}}}{m_\pi} \left(\frac{m_\mu}{m_\pi}\right)^2 I_3 + N^2 \left(\frac{m_\mu}{m_\pi}\right)^2 I_1 \Bigg] \mathrm{d}E_\pi \end{aligned} \quad (33)$$

where

$$\begin{aligned} I_1 &= \frac{\sqrt{E_\pi^2 - m_\pi^2}(w - E_\pi)^2}{m_\pi^2 (m_{\mathrm{K}}^2 - 2m_{\mathrm{K}} E_\pi + m_\pi^2)} \\ I_2 &= \frac{\sqrt{E^2 - m_\pi^2}(w - E_\pi)^2 (m_{\mathrm{K}} - E_\pi)^2}{m_\pi^2 (m_{\mathrm{K}}^2 - 2m_{\mathrm{K}} E_\pi + m_\pi^2)^2} \\ I_3 &= \frac{\sqrt{E^2 - m_\pi^2}(w - E_\pi)^2 (m_{\mathrm{K}} - E_\pi)}{m_\pi (m_{\mathrm{K}}^2 - 2m_{\mathrm{K}} E_\pi + m_\pi^2)^2} \\ I_4 &= \frac{\sqrt{E^2 - m_\pi^2}(w - E_\pi)^3 \{3(m_{\mathrm{K}} - E_\pi)^2 + (E_\pi^2 - m_\pi^2)\}}{m_\pi (m_{\mathrm{K}}^2 - 2m_{\mathrm{K}} E_\pi + m_\pi^2)^3}; \end{aligned}$$

w is the maximum energy of the pion, $w = (m_{\mathrm{K}}^2 - m_\mu^2 + m_\pi^2)/2m_{\mathrm{K}}$. Putting $m_\mu = 0$ we get the transition rate for the $\mathrm{K}^0 \to \pi^- + \mathrm{e}^+ + \nu$ with a fixed pion

energy. The comparison of the decay rates of $K_{\mu 3}$ and K_{e3} for the same pion energy would give the value of M/N as a function of pion energy.

In order to get possible information from the experiments available at present, let us take the following procedure. For the scalar variable (kp) on which the quantities M and N are dependent we substitute a constant (kq) where q_α is the energy-momentum four-vector of the pion in the corresponding $K(k) \to \pi(q)+\pi(k-q)$ decay. The range of variation of the ratio (kp) to (kq) is as follows:

$$0.98 \leqq \frac{(kp)}{(kq)} = \frac{2(m_K - E_\pi)}{m_K} \leqq 1.4 \quad \text{for} \quad K \to \pi+\mu+\nu,$$

$$0.92 \leqq \frac{(kp)}{(kq)} = \frac{2(m_K - E_\pi)}{m_K} \leqq 1.4 \quad \text{for} \quad K \to \pi+e+\nu.$$

As this variation is so small, the above substitution would be justified unless the quantities M and N have very peculiar steep dependence on the (kp). This will probably not be the case because in the present model the appearance of the massive baryons in the intermediate states will make the dependence less appreciable. Furthermore, the energy spectra of the pion, (33), in these $K_{\mu 3}$ and K_{e3} decays calculated on the assumption that M and N are constant exhibit broad maximum probability around the pion energy corresponding to the $K \to \pi+\pi$ decay.

So the above approximation seems to be a good one †. We could avoid this approximation by making a comparison between the decay rates of $K_1^0 \to \pi^+ + \pi^-$ and the rates of $K_{\mu 3}^0$ and K_{e3}^0 with the fixed secondary pion energy which corresponds to that of $K_1^0 \to \pi^+ + \pi^-$ decay instead of comparing the rates of $K_1^0 \to \pi^+ + \pi^-$, $K_{\mu 3}^0$ and K_{e3}^0 decays which will be discussed in the present paper. However, this is not feasible at present.

Under these simplifications we can easily evaluate the decay rates of the $K_{\mu 3}$ and K_{e3} decays,

$$W(K^0 \to \pi^- + \mu^+ + \nu) = \left(\frac{1}{2\pi}\right)^3 \left|\frac{f}{\sqrt{2}}\right|^2 (m_\pi)^5 \, 2 \frac{m_K}{m_\pi} (M^2 \times 0.15 + 2MN \times 0.021 + N^2 \times 0.036), \quad (34)$$

$$W(K^0 \to \pi^- + e^+ + \nu) = \left(\frac{1}{2\pi}\right)^3 \left|\frac{f}{\sqrt{2}}\right|^3 m_\pi{}^5 \frac{m_K}{m_\pi} M^2 \times 0.45. \quad (35)$$

Using (20) and (21) we have for the ratio of muon mode to electron mode

$$R = \frac{W(K_1^0 \to \pi^\mp + \mu^\pm + \nu)}{W(K_1^0 \to \pi^\mp + e^\pm + \nu)} = \frac{W(K_2^0 \to \pi^\mp + \mu^\pm + \nu)}{W(K_2^0 \to \pi^\mp + e^\pm + \nu)} = \frac{W(K^0 \to \pi^- + \mu^+ + \nu)}{W(K^0 \to \pi^- + e^+ + \nu)}$$

$$= 0.65 + 0.19 \frac{N}{M} + 0.16 \left(\frac{N}{M}\right)^2. \quad (36)$$

† In the problem of determining the types of Fermi interaction from the pion energy spectrum such a simplification might not be a good one. However, in the computation of the ratio of the decay rates it would not produce a serious error.

As remarked before, present preliminary experiments of K_2^0-meson indicate that $(K_2^0 \to \pi^{\mp}+\mu^{\pm}+\nu)/(K_2^0 \to \pi^{\mp}+e^{\pm}+\nu) \approx 1$. Also if we assume (25), $|\Delta I| = \frac{1}{2}$ for the strangeness non-conserving current, this value, $R \approx 1$, again follows from the experimental fact

$$W(K^+ \to \pi^0+e^++\nu) \approx W(K^+ \to \pi^0+\mu^++\nu).$$

Putting the value of (36) equal to one we find

$$N/M \approx 1.0 \quad \text{or} \quad -2.2 \tag{37}$$

i.e.

$$|N/M| \approx 1.$$

This result may not be so hard to understand. If we adopt scalar interactions, $\bar{\mu}(1+\gamma_5)\nu$ and $\bar{e}(1+\gamma_5)\nu$ for the interaction (15) and follow the same arguments as above, we get $R \approx 0.44$ (see ref.[13]), § 4). Also if we ignore the term N in the above computation we have $R \approx 0.65$. This would be a reflection of the fact that the phase space factor favours the K_{e3} decay by a factor ≈ 2 compared with the $K_{\mu 3}$-decay. Our arguments show that the presence of the term N which is of the same order of magnitude as the term M restores the ratio R to nearly unity. The lowest order perturbation calculation is also consistent with the above arguments [23]).

Now we proceed to the decay $K_1^0 \to \pi^+ + \pi^-$. We shall investigate the diagrams shown in fig. 4. At the vertex f the weak four-baryon interactions of

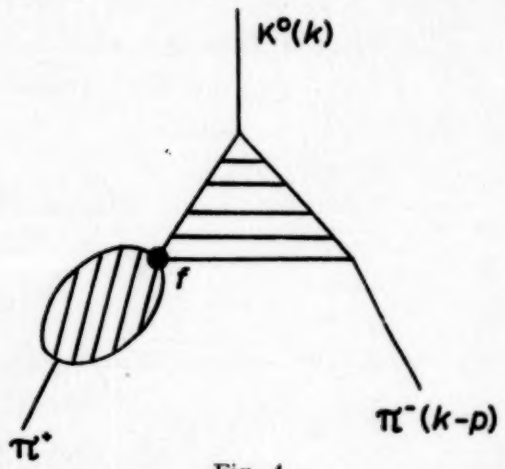

Fig. 4

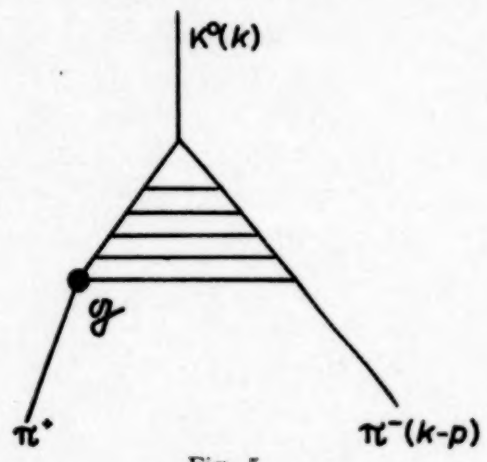

Fig. 5

the type (8) appear, and then the strangeness conserving part (black circle) of the interaction, $\mathcal{J}_\alpha^{(0)}$, transforms into a π^+-meson. Extending the discussion of § 1 which has enabled us to derive the effective interaction (9) from the basic interaction (4), we may replace the f-vertex by the effective interaction of the form,

$$\mathfrak{g}\mathcal{J}_\alpha^{(1)}\partial_\alpha\varphi_{\pi^+}^* \tag{38}$$

where $\mathfrak{g}$ is the effective coupling constant of the pion-two-fermion interactions which appeared in (9) and (12). Then the amplitude for the fig. 4 can

be rewritten in the form (see fig. 5)

$$\mathscr{M}^0(\pi) = \mathscr{M}^0(\mathrm{K}^0 \to \pi^+ + \pi^-) = \mathfrak{g}(\pi^-|\mathscr{J}_\alpha^{(1)}|\mathrm{K}^0)\partial_\alpha \varphi_{\pi^+}^*. \tag{39}$$

Taking this procedure, we are keeping a one-one correspondence between the triangular black boxes in fig. 3 and in fig. 4 or fig. 5. Fig. 5 is obtained from fig. 3 by replacing lepton vertex $f\bar{\nu}\gamma_\alpha(1+\gamma_5)\mu$ by $\mathfrak{g}\partial_\alpha\varphi_{\pi^+}^*$ (or, in momentum space, by $\mathfrak{g}p_\alpha\varphi_{\pi^+}^*(p)$). Then $\mathscr{M}^0(\pi)$ will be given by

$$\begin{aligned}\mathscr{M}^0(\pi) &= \mathfrak{g}\varphi_{\mathrm{K}}(k)\varphi_{\pi^-}^*(k-p)\varphi_{\pi^+}^*(p)\{(pk)M+(pp)N\} \\ &= -\,\mathfrak{g}\varphi_{\mathrm{K}}(k)\varphi_{\pi^-}^*(k-p)\varphi_{\pi^+}^*(p)\{\tfrac{1}{2}m_{\mathrm{K}}{}^2M+m_\pi{}^2N\};\end{aligned} \tag{40}$$

M and N are the same quantities as those which appear in (30). Now we are going to see how the decay rate of $\mathrm{K}_1{}^0 \to \pi^+ + \pi^-$ computed from the diagrams of the type of fig. 4, is related to those of K_{e3}^0 and $\mathrm{K}_{\mu 3}^0$-decay. From (40), we have

$$\begin{aligned}W(\mathrm{K}_1{}^0 \to \pi^+ + \pi^-) &= \tfrac{1}{8}\frac{\mathfrak{g}^2}{4\pi}m_\pi{}^2 \cdot m_\pi \frac{m_\pi}{m_{\mathrm{K}}}\left\{\tfrac{1}{2}\left(\frac{m_{\mathrm{K}}}{m_\pi}\right)^2 M+N\right\}^2\left\{1-4\left(\frac{m_\pi}{m_{\mathrm{K}}}\right)^2\right\}^{\frac{1}{2}} \\ &= \tfrac{1}{8}\frac{\mathfrak{g}^2}{4\pi}m_\pi{}^2 \cdot m_\pi(6.3M+N)^2\times 0.23.\end{aligned} \tag{41}$$

From (35) and (41) using the usual value of the Fermi coupling constant $f = 1.4\times 10^{-49}$ erg $\cdot$ cm^3 and the value of $\mathfrak{g}$, $(\mathfrak{g}^2/4\pi)m_\pi{}^2 \approx 1.8\times 10^{-15}$, we have

$$\begin{aligned}R' &= \frac{W(\mathrm{K}_1{}^0 \to \pi^\pm + \mu^\mp + \nu)}{W(\mathrm{K}_1{}^0 \to \pi^+ + \pi^-)} \approx \frac{W(\mathrm{K}_1{}^0 \to \pi^\pm + \mathrm{e}^\mp + \nu)}{W(\mathrm{K}_1{}^0 \to \pi^+ + \pi^-)} \\ &\approx 3.13\times\frac{M^2}{(6.3M+N)^2}.\end{aligned} \tag{42}$$

Inserting the values of M/N given by (37)

$$\begin{aligned}R' &= 6.0\times 10^{-2} \quad \text{for} \quad N/M \approx 1 \\ R' &= 1.9\times 10^{-1} \quad \text{for} \quad N/M \approx 2.2.\end{aligned} \tag{43}$$

This value should be compared with the experimental results (24), $R' \approx 3.1\times 10^{-4}$, based on the assumption (16) or the results (29), $R \approx 6.3\times 10^{-4}$, which are obtained by the stronger condition (25). The value of R' obtained seems to be too large by a factor $\approx 10^2$-10^3 compared with the experimental one †. Note that R' is not so sensitive to the value of N/M.

Now, we have to estimate the contribution of other diagrams which we have not taken into account in the above discussions. There seem to be two characteristic diagrams. We have started from a weak four-baryon interaction of the form,

† Compare with the discussions of ref.[14]), § 4, where the scalar type weak interaction is treated. The situation is worse than here as regards the branching ratio of $\mathrm{K}_1{}^0$-meson.

$$\frac{f}{\sqrt{2}} \mathcal{J}_\alpha^{*(1)} \mathcal{J}_\alpha^{(0)} + \text{h.c.} \tag{44}$$

One characteristic contribution is that the $\mathcal{J}_\alpha^{(0)}$ part transforms into a $\pi(1)$-meson and the $\mathcal{J}_\alpha^{(1)}$ part into a K and another $\pi(2)$-meson. The case $\pi(1) = \pi^+$ and $\pi(2) = \pi^-$ is the one which we have considered. There, the current $\mathcal{J}_\alpha^{(1)}$ is supposed to satisfy the condition (16), $\Delta S/\Delta Q = 1$. The case $\pi(1) = \pi^-$ and $\pi(2) = \pi^+$ could take place in higher order approximations † or even in the lowest order if we allow the current $\mathcal{J}_\alpha^{(1)}$ which satisfies $\Delta S/\Delta Q = -1$ from the outset. It is, however, very hard to imagine that the contributions of the latter diagrams are very different from the former ones. Rather, they may not be so appreciable since the available channels are less compared with the former cases as seen from the comparison of the channels (5) and (5′). The other characteristic contribution (which is not at all available for the lepton modes) arises when the $\mathcal{J}_\alpha^{(1)}$ part transforms only into a K-meson and the $\mathcal{J}_\alpha^{(0)}$ parts emit two outgoing pions, which is shown in fig. 6. Exactly in the same way as (38) we may

Fig. 6

approximate the transformation of $\mathcal{J}_\alpha^{(1)}$ into a K^0-meson † as the effective coupling of the form

$$g' \mathcal{J}_\alpha^{(0)} \partial_\alpha \varphi_{K^0} \tag{45}$$

where g' represents the contribution of the black circle. Then the matrix element of fig. 6 will take the form

$$g' \langle \pi | \mathcal{J}_\alpha^{(0)} | \pi \rangle \partial_\alpha \varphi_K . \tag{46}$$

Note that in this case the axial-vector (vector) part of the current $\mathcal{J}_\alpha^{(1)}$ is

† The V−A four-fermion interaction expressed in the charge exchange order (which may be regarded as the couplings between charged chiral currents as discussed in this paper) still preserve the V−A form in the charge retention order. It is not clear whether we need to introduce neutral chiral currents to (4) independently.

contributing if the K-meson is pseudoscalar (scalar). For the estimate of the order of magnitude of $\mathfrak{g}'$ we may employ the $K \to \mu+\nu$ decay. In fact, the matrix element of $K \to \mu+\nu$ shown in fig. 7 is given by

$$\mathfrak{g}'' j_\alpha^*(\mu, \nu)\partial_\alpha \varphi_K \tag{47}$$

where $\mathfrak{g}''$ denotes the contribution of black circle which represents the transformation of $\mathcal{J}_\alpha^{(1)}$ into a K^+-meson. From the partial decay life-time of $K^+ \to \mu^+ + \nu$, 2.08×10^{-8} sec, we can make an estimate

$$(\mathfrak{G}''^2/4\pi) m_\pi^2 = 1.08 \times 10^{-16}. \tag{48}$$

Fig. 7

This is smaller by about a factor ≈ 10 than the value of $\mathfrak{g}^2$ of (38). $\mathfrak{g}'$ will be of the same order of magnitude as $\mathfrak{g}''$. On the other hand, the contribution of triangular black box of fig. 4, $|\langle \pi^- | \mathcal{J}_\alpha^{(1)} | K^0 \rangle|^2$ might be smaller than that of fig. 6, $|\langle \pi | \mathcal{J}_\alpha^{(0)} | \pi \rangle|^2$, since the K-baryon coupling constant seems to be smaller than that of π-baryon interactions [19]). Altogether, the contribution of fig. 6 might be as important as that of fig. 4.

Summing up, even if we assume that all these three contributions are of the same order of magnitude and, furthermore, that they are all *coherent* with each other, we may only hope to increase the rate of $K_1^0 \to \pi^+ + \pi^-$ by an order of magnitude, unless renormalization effects on these characteristic diagrams, which are here neglected, are very important †. Therefore, according to the estimate (43), the decay rates of K_{e3}^0 and $K_{\mu 3}^0$ are still larger than the experimental figures by a factor $10\text{-}10^2$. It is interesting to apply the same sort of argument to the branching ratio of K^+-decay. We have

$$R'' = \frac{W(K^+ \to \pi^0 + e^+ + \nu)}{W(K^+ \to \pi^+ + \pi^0)} \approx \frac{W(K^+ \to \pi^0 + \mu^+ + \nu)}{W(K^+ \to \pi^+ + \pi^0)} \approx 2R'. \tag{49}$$

As is the case with the K^0-meson, there is an uncertainty of about an order of magnitude concerning this ratio.

† Perhaps we may believe this from the facts discussed in § 1, (b) and (c), except for the effect of final state interactions of $K_1^0 \to \pi^+ + \pi^-$ decay.

Now, we shall draw some possible conclusions from the results obtained above. The conclusion would depend on the effects of final state interactions which might be important in the $K \to 2\pi$ decays. Therefore, we shall discuss two different possibilities separately.

(A) *The final state interactions are not important in the* $K \to \pi+\pi$ *decays*

(a) If this is the case we are inclined to conclude that *the coupling constant* f^2 *of the strangeness non-conserving baryon-lepton part* (15) *of the interaction* (4) *is smaller than the square of the usual Fermi coupling constant by, at least, an order of magnitude.* In this case we conclude that the ratio R'' of the rates of K^+_{e3} and $K^+_{\mu 3}$ to that of $K^+ \to \pi^+ + \pi^0$ obtained in (49) will not have much meaning although the result is not far removed from the experimental one. The reason is as follows:

If final state interactions are not significant, the frequency ratio of the $K^+ \to \pi^+ + \pi^0$ to $K_1{}^0 \to \pi^+ + \pi^-$ is very hard to understand unless there is a selection rule. It should be noted that the frequency ratio of $K^+ \to \pi^+ + \pi^- + \pi^+$ to $K_1{}^0 \to \pi^+ + \pi^-$ can be explained by the argument of available phase space volume [23]). Therefore, we should, most probably, expect some selection rule which reduces the probability of the $K^+ \to \pi^+ + \pi^0$ compared with the $K_1{}^0 \to \pi^+ + \pi^-$. We may imagine such a situation that for the $K \to \pi+\pi$ decays the $|\Delta I| = \frac{1}{2}$ part of the interaction is, in effect, somehow enhanced and smaller components of $|\Delta I| = \frac{3}{2}$ and $\frac{5}{2}$ (perhaps of the order 10 percent) come into play. Such a selection rule will not exist for the leptonic decays of K-meson. Indeed, the fact that the decay rates of $K^+_{\mu 3}$, K^+_{e3}, $K^0_{\mu 3}$ and K^0_{e3} are very close to each other will strongly suggest that this is the case. Therefore, our arguments which take the result of the branching ratio of the K^0-meson (43) seriously will be justified since the $K_1{}^0 \to \pi^+ + \pi^-$ decay is most free from having such a selection rule in isobaric spin space.

Then, if we want to keep the simple structure of the assumed form of universal interaction (4) with (6), one of the simplest and most straight-forward solutions is to assume that the squared coupling constant of the strangeness non-conserving baryon-lepton part of the interaction (4) is smaller than the usual Fermi coupling constant $(1.4 \times 10^{-49}\ \mathrm{erg \cdot cm^3})^2$ by, at least, one order of magnitude. This predicts that the decay rates of $\Lambda^0 \to p + e^- + \bar{\nu}$ and $\Sigma^- \to n + e^- + \bar{\nu}$ etc. are smaller than the values, (13) and (14), expected on the basis of the usual universal Fermi interactions by, at least, an order of magnitude. This is, indeed, suggested by the present experimental information. In other words, the present experimental information on the rates of hyperon decays into leptons would not be striking at all from the consideration of the branching ratios of the K^0-meson.

Therefore, the above discussion may be regarded as a supporting evidence on the standpoint II discussed in § 1.

(b) We might avoid the above conclusion if we abandon the assumed simple structure of the universal interaction, that is, the self-coupling (4) of the chiral current given by (6). Indeed, the result (43) follows from this structure which enables us to draw a one-one correspondence between the triangular black boxes in fig. 3 and fig. 4. Perhaps, we should try for the $K_{\mu 3}$ and K_{e3} decays to decrease the number of channels available or to have destructive interference between them which appear in the interaction (15). On the contrary, we must have various coherent channels for the strangeness non-conserving four-baryon interactions (44) in order to retain a large probability for the $K_1^0 \to \pi^+ + \pi^-$. Then, we might be able to preserve the universality of the absolute value of the Fermi coupling constant. A similar situation exists for the pion decays. If the V—A Fermi interaction is completely symmetrical with respect to an electron and a μ-meson, the frequency ratio of pion decays $W(\pi \to e+\nu)/W(\pi \to \mu+\nu) \approx 1.36 \times 10^{-4}$ is predicted. In order to get a smaller ratio which seems to be indicated by experiments [24]) we may think that the complete symmetry is not correct and there exist some destructive interference for the electron channels although we may still be able to keep the universality of the absolute value of the coupling constant [25]). We might imagine such a possibility in our case also. However, the situation does not seem to be easy, since we have, after all, to explain the small rates of hyperon decays into leptons with the usual Fermi coupling constant.

Added in proof: The small branching ratios of leptonic modes of K_1^0-meson decay given by (24) or (28) are based on the assumption that the strangeness non-conserving currents, $\mathcal{J}_a^{(1)}$, satisfy $\Delta S/\Delta Q = 1$. If we invoke the currents of the type (5) which satisfy $\Delta S/\Delta Q = -1$, the relations (20) and (21) would be violated. Then it may be possible to construct models of Fermi interactions in which $W(K_1^0 \to e^{\pm}+\nu+\pi^{\mp}) \gg W(K_2^0 \to e^{\pm}+\nu+\pi^{\mp}) \approx (3-7) \times 10^6$ sec^{-1} and $W(K_1^0 \to \mu^{\pm}+\nu+\pi^{\mp}) \gg W(K_2^0 \to \mu^{\pm}+\nu+\pi^{\mp}) \approx (3-7) \times 10^6$ sec^{-1}.

According to our estimate (43), we would then expect the $K_1^0 \to e^{\pm}+\nu+\pi^{\mp}$ and $K_1^0 \to \mu^{\pm}+\nu+\pi^{\mp}$ in, at least, a few per cent of the K_1^0-decays. Such a possibility may not be completely ruled out by the present experimental evidence. This might be a way to preserve the universality of the coupling constant. For this case, however, we must expect the occurrence of the reactions satisfying $|\Delta S| = 2$ such as $\Xi^- \to n+\pi^-$ decay.

(B) *The final state interactions are very important in the* $K \to \pi+\pi$ *decays*

In this case we may hope that the final state interactions may enhance the probability of the $K_1^0 \to \pi^+ + \pi^-$ decay and, therefore, the branching ratios of $K_{\mu 3}^0$ and K_{e3}^0 to $K_1^0 \to \pi^+ + \pi^-$ may be explained. It is interesting to notice the following two facts. (i) By enhancing the probability of $K_1^0 \to$

$\pi^+ + \pi^-$ we may at the same time also explain the experimental frequency ratio of $K^+ \to \pi^+ + \pi^0$ to $K_1^0 \to \pi^+ + \pi^-$. (ii) The branching ratio of K^+-meson decays, as shown by (49), is not incompatible with the experiments. This theoretical ratio may be more reliable in the present case because we are ascribing the frequency ratio of $K^+ \to \pi^+ + \pi^0$ to $K_1^0 \to \pi^+ + \pi^-$ not to the selection rule which may work on the $K^+ \to \pi^+ + \pi^0$ decay but to the final state interactions. Therefore, the enhancement of the probability of $K_1^0 \to \pi^+ + \pi^-$ by final state interactions may not be a bad idea since, then, the simultaneous explanations of the ratios, $W(K^+ \to \pi^+ + \pi^0)/W(K_1^0 \to \pi^+ + \pi^-)$, $W(K_{e3}^+, K_{\mu 3}^+)/W(K^+ \to \pi^+ + \pi^0)$ and $W(K_{e3}^0, K_{\mu 3}^0)/W(K_1^0 \to \pi^+ + \pi^-)$ might be possible.

Okubo and Marshak [26]) suggest the possibility of having a strong final state interaction between the π^+ and π^- in the $I = 0$ state. This standpoint, of course, requires that the weak interaction must have an appreciable component of $|\Delta I| = \frac{3}{2}$. It would probably be wise to keep the above possibility in mind since we might be able to retain the simple structure of the interaction (4). However, there are many problems to be solved. One of them is the branching ratio of $K_1^0 \to \pi^0 + \pi^0$ to $K_1^0 \to \pi^+ + \pi^-$. Quantitatively we may have to look for the interference effects of the state $I = 0$ with other states. Furthermore, the decay rates of hyperon decays into leptons have to be explained.

3. Phenomenological estimate † of the Decay Rates of $\Lambda^\circ \to p + e^-(\mu^-) + \bar{\nu}$

The comparatively small rates of hyperon decays into leptons recently reported seem to pose a problem for the universal V—A four-fermion interactions. We know that the $K \to \mu + \nu$ and $K \to \pi + \mu(e) + \nu$ interactions exist. Then the hyperon decays into leptons should, at least, take place through these interactions even if we do not introduce direct interactions between baryons and leptons. The computation of such a mechanism may give some information on the lower limit of the rates of these decays. It is interesting to compare this lower limit with experimental observations in order to study the consistency of the theory.

3.1 THE $\Lambda^\circ \to p + \mu^- + \bar{\nu}$ DECAY THROUGH $K \to \mu + \nu$ INTERACTION

Through the virtual processes shown in fig. 8, the $\Lambda^0 \to p + \mu^- + \bar{\nu}$ will take place whatever the true mechanism of this decay will be ††. For the effective

† The author wishes to thank Professor R. E. Marshak for raising the problem discussed in this section.

†† As a matter of fact, the idea to take the $K \to \mu + \nu$ interaction as a primary agency for deriving the $\Lambda^0 \to p + \mu^- + \bar{\nu}$ would not be correct, since it cannot account for the $K \to \mu + \nu + \pi$ decay [7]).

interaction we take the following two possibilities:

$$G_{\mathrm{K}}(\bar{\Lambda}^0\gamma_5 \mathrm{pK}^{*+}+\bar{\Lambda}^0\gamma_5 \mathrm{nK}^{*0})+\mathrm{h.c.} \tag{50}$$

and

$$G'_{\mathrm{K}}(\bar{\Lambda}^0 \mathrm{pK}^{*+}+\bar{\Lambda}^0 \mathrm{nK}^{*0})+\mathrm{h.c.} \tag{51}$$

where G_{K} and G'_{K} are the effective coupling constants. Let us assume that the interaction responsible for the $\mathrm{K} \to \mu+\nu$ interaction is a local interaction of the form

$$\bar{\mu}\gamma_\alpha(1+\gamma_5)\nu\partial_\alpha\varphi_{\mathrm{K}} \quad \text{or} \quad \bar{\mu}(1+\gamma_5)\nu\varphi_{\mathrm{K}}.$$

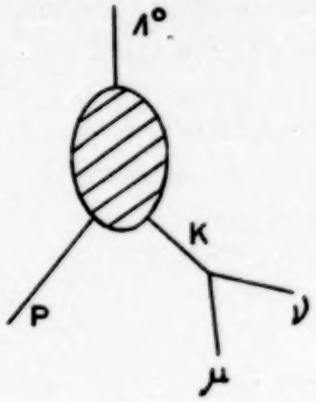

Fig. 8

This has already been discussed in connection with the universality of the weak boson-two-fermion interactions [14, 27]). The only change we should make now is to replace the life-time of the $\mathrm{K} \to \mu + \nu$ adopted there, 1.4×10^{-9} sec, by the present experimental one, 2.08×10^{-8} sec. Then we get for the rate

$$W(\Lambda^0 \to \mathrm{p}+\mu^-+\bar{\nu}) \approx (G_{\mathrm{K}}^2/4\pi)\times 2.2\times 10^4 \ \mathrm{sec}^{-1} \tag{52}$$

or

$$W(\Lambda^0 \to \mathrm{p}+\mu^-+\bar{\nu}) \approx (G_{\mathrm{K}}'^2/4\pi)\times 2.8\times 10^5 \ \mathrm{sec}^{-1}. \tag{53}$$

For the usual universal V—A Fermi interaction we have [14, 15])

$$W(\Lambda^0 \to \mathrm{p}+\mu^-+\bar{\nu}) \approx 1\times 10^7 \ \mathrm{sec}^{-1}. \tag{54}$$

For the value $G_{\mathrm{K}}^2/4\pi \approx G_{\mathrm{K}}'^2/4\pi \approx 1$, (52) and (53) are smaller than (54) by a factor $40 \approx 500$. This is compatible with the present experimental results. It may be noted that even if the effective coupling constant G_{K} of the pseudoscalar K-N-Λ interaction is as large as that of pion-nucleon interaction (universal strong pseudo-scalar baryon-meson interaction) the decay rate is smaller than the value (54) by a factor ≈ 30 and is still consistent with the present experiment. The $\mathrm{K} \to \mu+\nu$ decay is known to have an unusually small decay rate. That is, if we assume the same coupling constant for both the $\pi \to \mu+\nu$ and $\mathrm{K} \to \mu+\nu$ decay the $\mathrm{K} \to \mu+\nu$ could have a life-time $\approx 1.4\times10^{-9}$ sec [14]), which is about 20 times shorter than the observed one. This would be one of the reasons for the smallness of the decay

rate obtained above of $\Lambda^0 \to p+\mu^-+\bar{\nu}$ expressed in terms of the observed decay rate of $K \to \mu+\nu$. In the model of universal Fermi interaction, the slowness of the $K \to \mu+\nu$ decay may have an origin either in the possibility that the coupling constant G_K of the K-$\mathcal{N}$-$\Lambda(\Sigma)$ interaction is smaller than that of the pion-baryon coupling constant or in that the coupling constant of the $\Lambda(\Sigma) \to \mathcal{N}+\mu+\nu$ interaction is smaller than the usual Fermi coupling constant (standpoint (II) discussed in § 1).

3.2. THE $\Lambda \to p+e^-(\mu^-)+\bar{\nu}$ DECAY THROUGH THE $K \to e(\mu)+\nu+\pi$ INTERACTION

The foregoing discussion cannot be used for the $\Lambda^0 \to p+e^-+\bar{\nu}$ decay since we do not know the exact decay rate of $K \to e+\nu$. So we should relate the $K \to e+\nu+\pi$ to the $\Lambda^0 \to p+e^-+\bar{\nu}$ decay. In fact, if we adopt a derivative type $K \to e+\nu$ interaction which is a faithful reflection of the V—A Fermi interaction we get [14]) a probability for the $\Lambda^0 \to p+e^-+\bar{\nu}$ which is about 10^{-4} times smaller than that of $\Lambda^0 \to p+\mu^-+\bar{\nu}$ obtained in (52) or (53). As mentioned in § 2, different mechanisms are at work for the $K_{\mu 2}$, K_{e2} and $K_{\mu 3}$, K_{e3} decays. Therefore, in applying the same sort of phenomenological discussion to the $\Lambda^0 \to p+e^-+\bar{\nu}$, the $K \to e+\nu+\pi$ interaction should be regarded as an important agency. It would, perhaps, be unwise to perform a perturbation calculation. Rather we want to use the known rate of $\Lambda^0 \to p+\pi^-$ as an experimental calibration of the complicated effects. In fact, we may have an estimate of the decay rate of $\Lambda^0 \to p+e^-+\bar{\nu}$ through the interactions, $K^0 \to e+\nu+\pi$ and $K^+ \to e+\nu+\pi$, by comparing with the rate of $\Lambda^0 \to p+\pi^-$ through the interaction $K^0 \to \pi^+ + \pi^-$ (the contribution of the interaction $K^+ \to \pi^+ + \pi^0$ would be negligible). The typical diagrams are shown in fig. 9 and fig. 10. First, let us try the simplest case where the

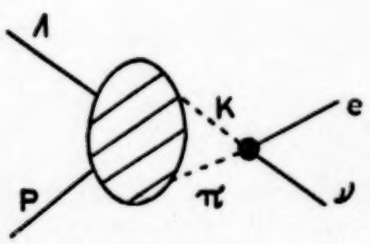

Fig. 9

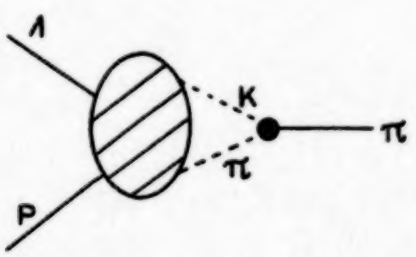

Fig. 10

lepton vertex is scalar with respect to rotation. We shall assume the interaction form of $K \to e+\nu+\pi$

$$h\varphi_K \varphi_\pi^* \bar{\nu}(1-\gamma_5)e+\text{h.c.} \tag{55}$$

For the $K \to \pi+\pi$ interaction, we take the form

$$g\varphi_K \varphi^*{}_\pi \varphi_\pi + \text{h.c.} \tag{56}$$

The values of the coupling constants, h and g, will be determined from the

known rates of the $K \to e+\nu+\pi$ and $K \to \pi+\pi$ decays. After using the equation of motion of baryons, the matrix elements of fig. 9 and fig. 10 will be reduced to the forms

$$hM\bar{p}\Lambda^0\bar{e}(1+\gamma_5)\nu \tag{57}$$

and

$$gM'\bar{p}\Lambda^0\varphi_\pi. \tag{58}$$

We have assumed that the K-meson is pseudoscalar † and the baryons have the same relative parity. M and M' denote the contributions of black boxes and are the same functions of the energy of the secondary proton. However, as the range of variation of the proton energy of the $\Lambda^0 \to p+e^-+\bar{\nu}$ decay is quite small compared with the energy of proton of $\Lambda^0 \to p+\pi^-$ we can take $M \approx M'$. From (57) and (58) we can compute the branching ratio of the Λ^0-decay using, for instance, the branching ratio of K^0-meson given by (24). In this ratio the unknown quantities M's are cancelled out, and

$$\frac{W(\Lambda^0 \to p+e^-+\bar{\nu})}{W(\Lambda^0 \to p+\pi^-)} \approx 8.0\times 10^{-6}. \tag{59}$$

Here we have neglected the contribution of $K^+ \to \pi^+ + \pi^0$ and assumed that the interaction $K^0 \to \pi+e+\nu$ and $K^+ \to \pi+e+\nu$ give the same and *coherent* contributions.

The above case may not be realistic and we may have to study the case in which the lepton vertex is given by a vector type as is suggested by the rarity of the mode $K \to e+\nu$ compared with that of $K \to \mu+\nu$. So we may start from the effective hamiltonian for the $K \to e+\nu+\pi$ given by (32),

$$h\varphi_K(k)\varphi_\pi^*(k-p)\bar{\nu}(\gamma k)(1+\gamma_5)e+\text{h.c.} \tag{60}$$

Corresponding to (60) we write the interaction $K^0 \to \pi^+ + \pi^-$ in the form

$$g\varphi_K(k)\varphi_\pi^*(k-p)(pk)\varphi_\pi(p)+\text{h.c.}; \tag{61}$$

h and g are again the effective coupling constants. Then the matrix elements of fig. 9 and fig. 10 will, in general, take the following forms:

$$h\bar{p}\left(M\gamma_\alpha+N\frac{k_\alpha}{Q}+\ldots\right)\Lambda^0\bar{e}\gamma_\alpha(1+\gamma_5)\nu$$

and

$$g\bar{p}\left(M'\gamma_\alpha+N'\frac{k_\alpha}{Q}+\ldots\right)\Lambda^0 p_\alpha\varphi_\pi(p).$$

The brackets may include all the terms which could be induced and are allowed from the general covariance argument. Q, for instance, is a quantity which has the dimension of mass (probably of the order of the masses of baryons which appear in the intermediate state). Again M and N etc. are functions of the secondary proton energy and we can put $M \approx M'$, $N \approx N'$

† The scalar K-meson gives nearly the same result.

and so on as we have done in computing (57) and (58). In the following estimate we only retain the first term within the brackets. We do not have a reliable knowledge about the contribution of other terms. Perhaps some of them should not be discarded. However, we may hope that the neglect of these terms is not so significant in computing the ratio of two similar processes (not the rate of individual process). Exactly in the same way as we obtain (59), we arrive at

$$\frac{W(\Lambda^0 \to \mathrm{p}+\mathrm{e}^- +\bar{\nu})}{W(\Lambda^0 \to \mathrm{p}+\pi^-)} \approx 3.0\times 10^{-4}. \tag{62}$$

This value is smaller by a factor ≈ 70 than the one predicted by the V−A Fermi interaction (see (13)) and seems to exhibit the fact that the present experimental rates of $\Lambda^0 \to \mathrm{p}+\mathrm{e}^-(\mu^-)+\bar{\nu}$ are reasonable from theoretical point of view. It is quite interesting to see whether the accumulation of more data of these events will be consistent with the branching ratio obtained in (62).

Combining the results of § 2 and § 3 we would like to emphasize the fact that the experimentally indicated small probability of the hyperon decays into leptons is rather desired from the branching ratios of the K^0-meson.

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INFLUENCE OF THE FINITE DIMENSIONS OF THE NUCLEUS ON THE RELATIVE CONVERSION COEFFICIENTS IN THE L-SUBSHELLS †

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Abstract: Measurements have been made of the relative internal conversion coefficients in the L-subshells for three pure M1 transitions: 46.5 keV in Bi^{210}, and 115.1 and 238.6 keV in Bi^{212}. It is shown that in order to obtain agreement with the experimental data, it is necessary to take into consideration the finite dimensions of the nucleus in the theoretical calculations of the L internal conversion coefficients.

Measurements have also been made of $L_I : L_{II} : L_{III}$ for the 277.3 keV M1 transition in Pb^{208}.

1. Introduction

It has been established in several experimental studies [1–6]) that K conversion coefficients for M1 transitions determined experimentally for heavy nuclei are much smaller than the theoretical values computed by Rose [7]) for a point nucleus. The difference is approximately equal to the correction (≈ 30 %) for the finite dimensions of the nucleus proposed by Sliv [8]). It has been shown theoretically [9]) and experimentally [1, 4, 10]) that approximately an equal correction must also be made in the L_I conversion coefficients. It may thus be considered as proven that in theoretical calculations of the internal conversion coefficients in the K and L_I shells for M1 transitions the finite dimensions of the nucleus must be taken into account.

As for the influence of the finite dimensions of the nucleus on conversion in the L_{II} shell, the available experimental data [1, 4, 10, 11]) are contradictory, due to the poor precision of the measurements. Band and Sliv recently computed the conversion coefficients in the L_I, L_{II} and L_{III} [12]), subshells. The preliminary results of these computations kindly communicated to us by the authors, show that the influence of the finite dimensions of the nucleus is essentially different for the different L subshells. The correction is weakly dependent on the energy, and for $Z = 83$ comprises ≈ 33 % in L_I,

† Paper read at the Eighth Annual Symposium on Nuclear Spectroscopy of the USSR Academy of Sciences, January 1958, Leningrad.

$\approx$ 11 % in L_{II}, and is small in L_{III}. The ratio $L_I : L_{II} : L_{III}$ should therefore be sensitive to the correction for the finite dimensions of the nucleus.

The aim of the present communication is to verify experimentally the influence of the finite dimensions of the nucleus on the relative conversion coefficients in the L subshells for M1 transitions. The following points are essential for such a verification: 1) the accuracy of measurement of the relative intensities must be at least 5 %; 2) it must be certain that the lines investigated are pure M1 transitions. Even a slight admixture of E2 can strongly affect the ratio $L_I : L_{II} : L_{III}$.

2. Experimental Data

Internal conversion in the L subshells was studied for the following three M1 transitions: 46.5 keV in the decay RaD $\xrightarrow{\beta}$ RaE (Bi^{210}_{83}) and 115.1 and

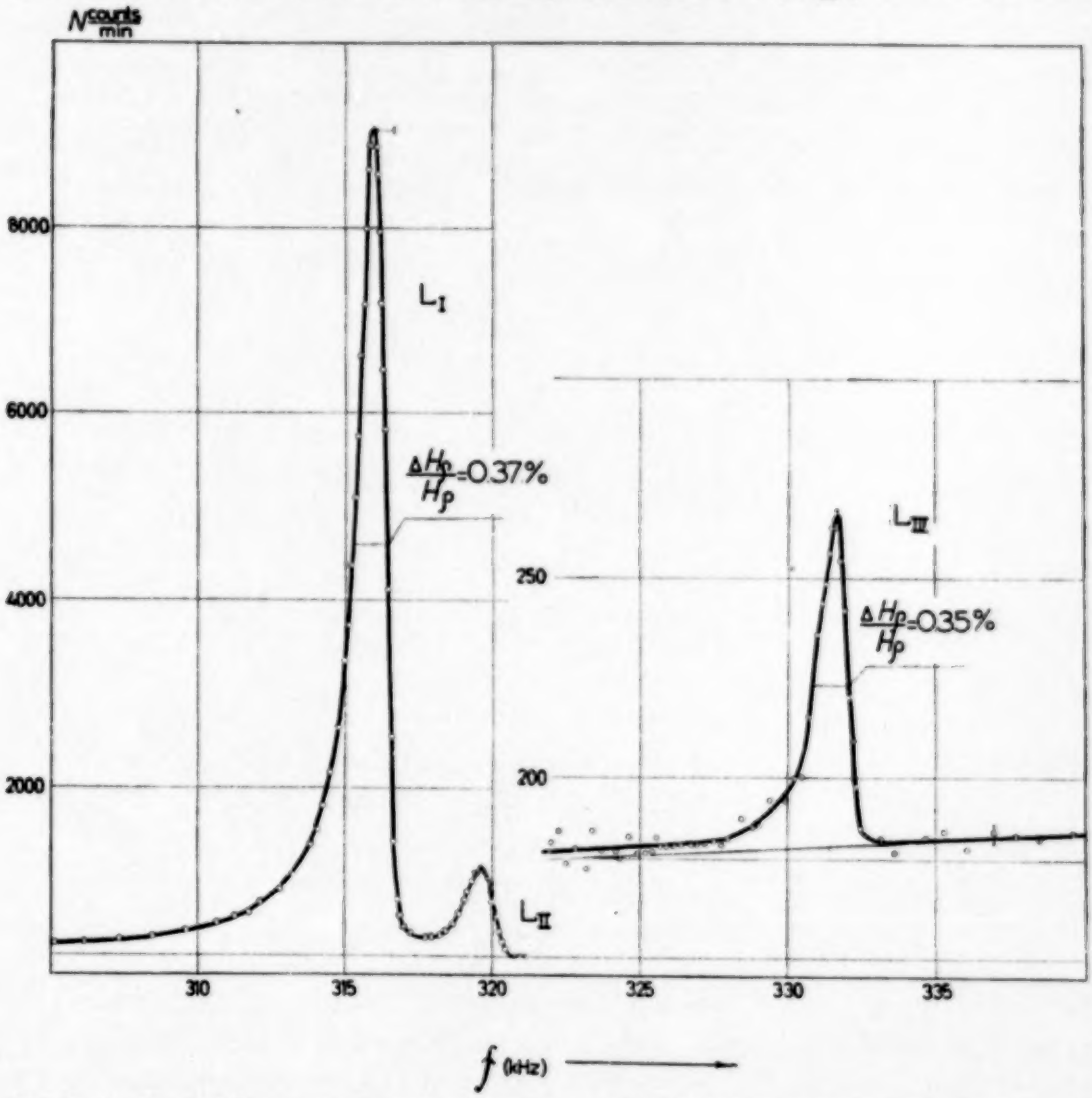

Fig. 1. L_I, L_{II} and L_{III} are conversion lines for the 46.5 keV transition in Bi^{210}. On this and the following graphs the frequency of the proton resonance f, proportional to $H\rho$, is plotted on the axis of abscissae.

238.6 keV in the decay ThB $\xrightarrow{\beta}$ ThC (Bi^{212}_{83}). According to ref.[13]), all three transitions take place between levels of spin 0 and 1, and are therefore pure M1 transitions.

The experiment was performed on a caetron type magnetic spectrometer[14]).

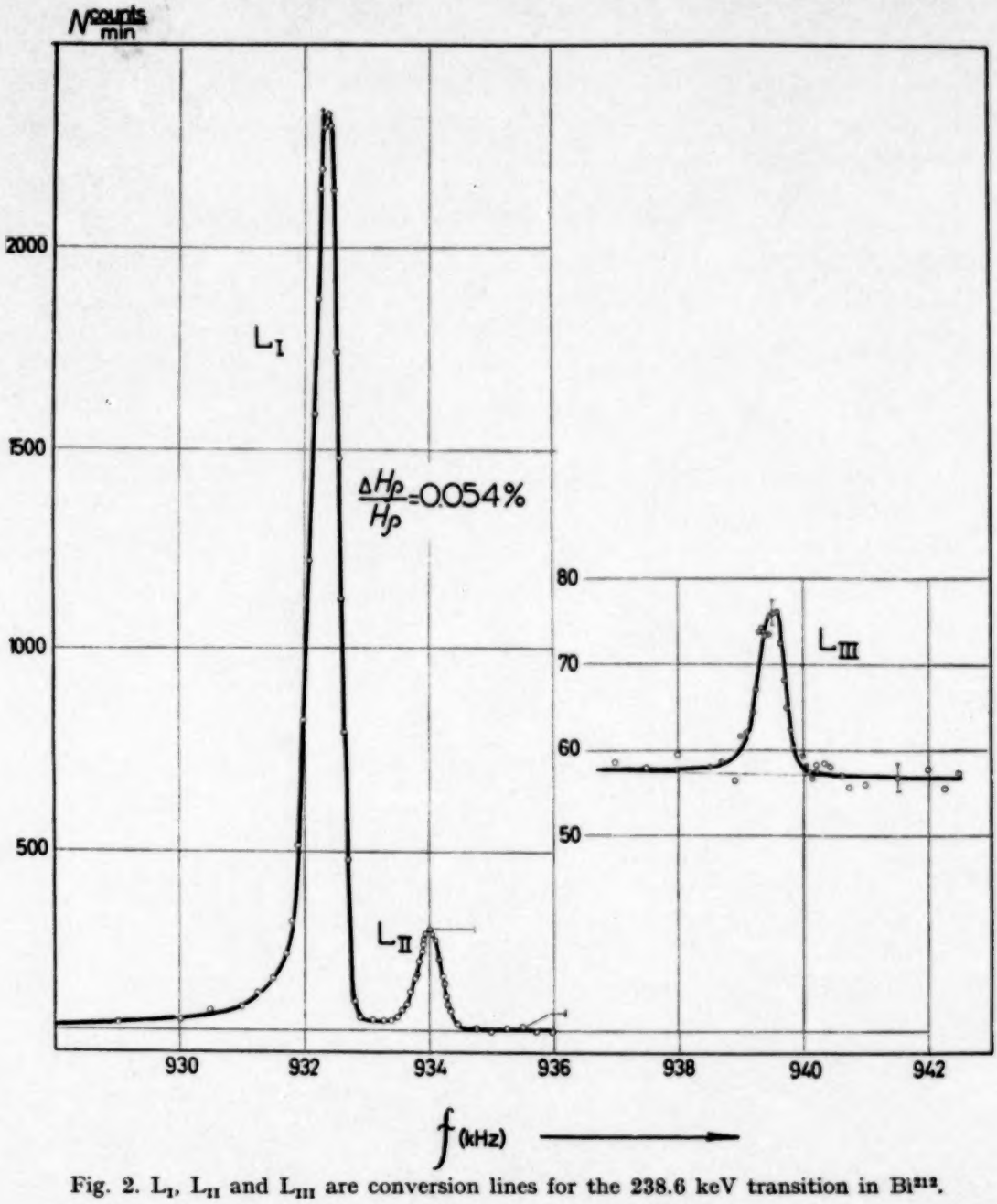

Fig. 2. L_I, L_{II} and L_{III} are conversion lines for the 238.6 keV transition in Bi^{212}.

Measurement and stabilization of the magnetic field were carried out by the proton magnetic resonance method [15]). The electrons were registered with Geiger-Müller counters. The resolving power of the apparatus was varied from 0.05 to 0.36 %, and for each line was chosen so that at maximum relative aperture complete separation of the lines was achieved. Figs. 1—4

represent several series of measurements of the above-mentioned lines. The 115.1 keV transition is rather weak; the ratio $L_I : L_{II}$ for this transition was therefore measured at a resolution of 0.14 % (fig. 3), and the ratio $L_I : L_{III}$ at a resolution of 0.23 % (fig. 4). The L_{III} line for this transition has not been observed heretofore.

The ratio $L_I : L_{II} : L_{III}$ was determined directly from the heights of the corresponding lines. In treating the experimental data we introduced, in addition to the correction for decay, the following corrections: 1) a correction

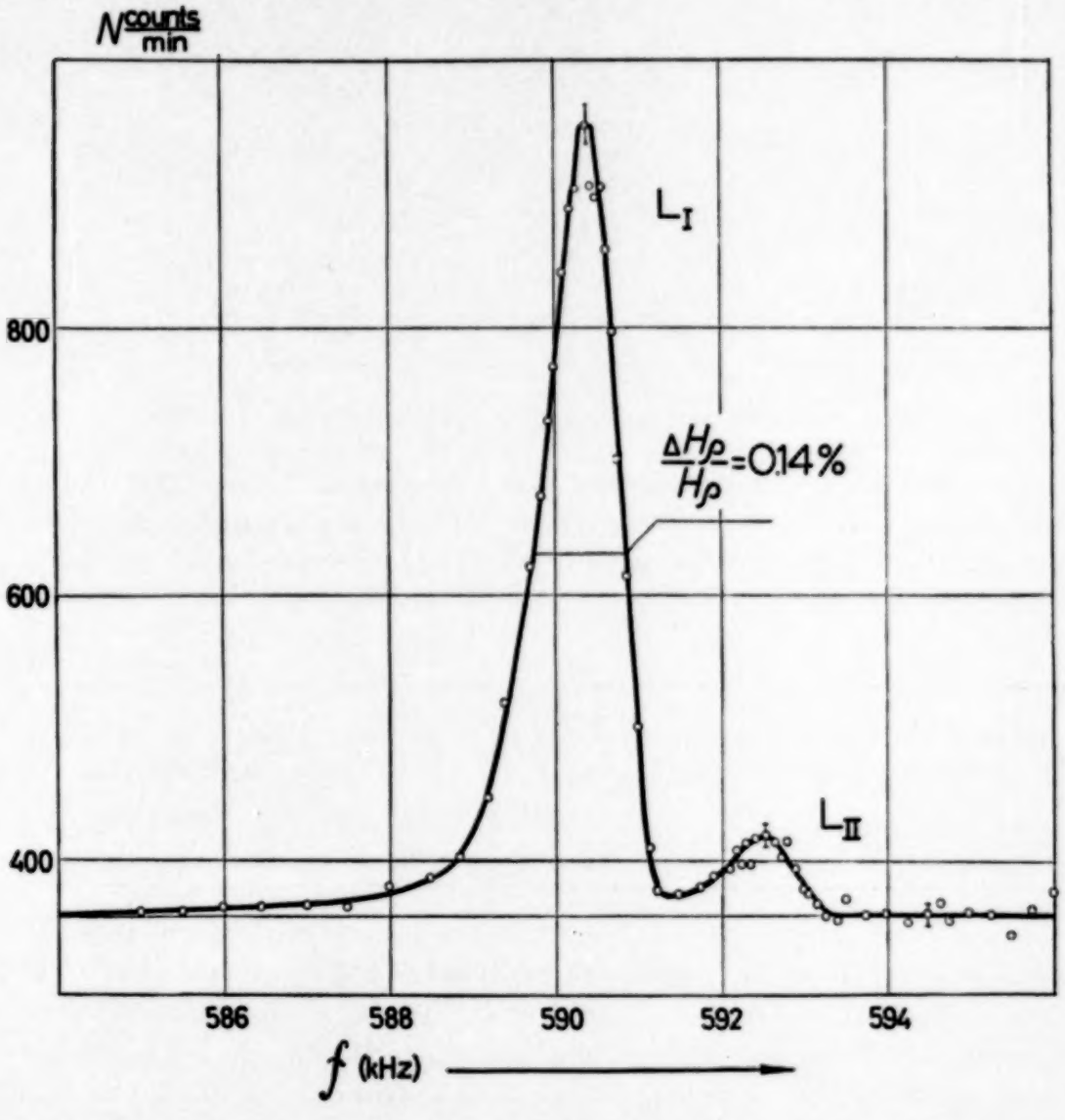

Fig. 3. L_I and L_{II} are conversion lines for the 115.1 keV transition in Bi^{212}.

of $1 \div 2$ % for the length of the tail of the L_{II} line under the L_I line; 2) for the 46.5 keV transition a correction was made, in addition, for the variation of the half-width of the line with $H\rho$. This dependence of the half-width on $H\rho$ in our apparatus is to be explained by the distortion of the topography of

the field at $H\rho < 1000$ oersted · cm and was investigated on the conversion spectrum of an active deposit of radiothorium [16]). The correction comprises 1.2 % for $L_I : L_{II}$ and 6.5 % for $L_I : L_{III}$. Corrections were not made for absorption in the films of the counters or for missed counts; these were small in our conditions (< 1 %).

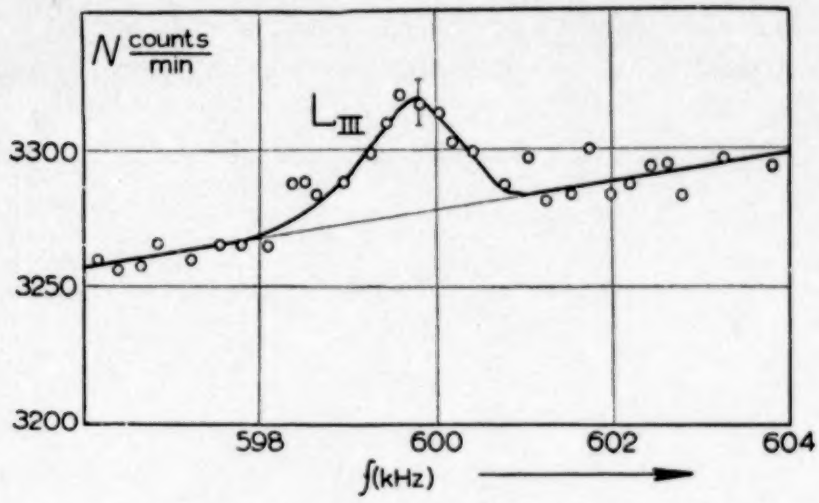

Fig. 4. L_{III} is conversion line for the 115.1 keV transition in Bi^{212}.

Four-five series of measurements were made for each line. Table 1 gives the average values. The errors given in the table are slightly greater than the statistical ones, since the possible effect of systematic errors was taken into consideration.

TABLE 1

Decay	E_γ keV	$L_I : L_{II} : L_{III}$
RaD → RaE (Bi^{210}_{83})	46.5	100 : (10.6±0.2) : (0.93±0.05)
ThB → ThC (Bi^{212}_{83})	115.1	100 : (10.4±0.3) : (0.88±0.10)
— " —	238.6	100 : (10.4±0.2) : (0.74±0.05)

The relative conversion coefficients in the L subshells for the 46.5 and 238.6 keV transitions have been repeatedly measured by other workers. Our data differ markedly from the results of the investigations performed prior to 1955. The divergence of the results amounts to 100 %. The large errors in these papers may be due either to insufficient resolution, or to the application of the photographic method of recording. For this reason in table 2 we compare our data only with more recent works [10,11,17,18,19]). (In ref.[18]) a special method was employed by which it was possible to eliminate the errors connected with the photographic recording of the electrons.) It is evident from table 2 that within the limits of experimental error our data agree well with those of the other authors for both transitions; only for the

ratio $L_I : L_{II}$ is there a divergence of $\approx 10\ \%$ with ref.[11]). In this investigation, which was performed at our laboratory, the errors indicated in the table are purely statistical and do not include the systematic errors which we

TABLE 2

Transition	$L_I : L_{II} : L_{III}$	Reference
46.5 keV	100 : (10.2±0.7) : (0.95±0.05)	[17])
	100 : (10.5±0.4) : (1.0 ±0.1)	[18])
238.6 keV	100 : (10.8±0.1) : (0.80±0.02)	[19])
	100 : (11.6±0.3) : (0.66±0.04)	[11])
	100 : (10.0±0.7)	[10])

discovered later. By introducing the appropriate correction, agreement is obtained between the results.

It should be observed that in ref.[20]) in taking down the dispersion spectrum of RaD a line of 31.9 keV and intensity comparable to that of L_{III} was discovered between the lines L_{II} and L_{III}. As appears from fig. 1, we discovered no such line in this region.

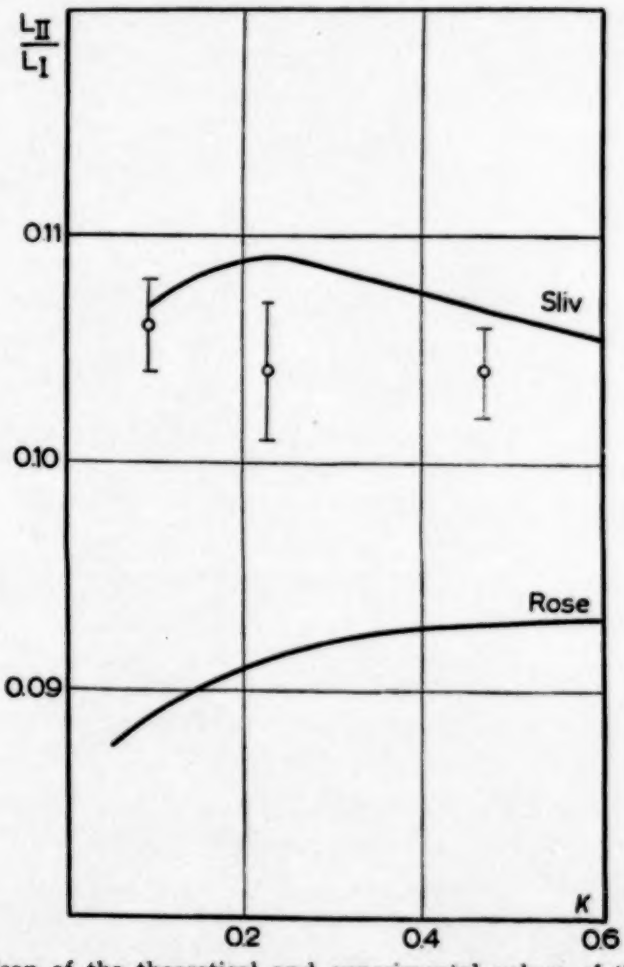

Fig. 5. Comparison of the theoretical and experimental values of the ratio $L_{II} : L_I$.

3. Comparison with the Theory

In order to compare the experimental data with the theoretical ones in figs. 5 and 6 the theoretical ratios $L_{II} : L_I$ and $L_{III} : L_I$ have been plotted against the energy. The experimental values of these ratios are given in the same graph. The lower curves on both graphs were plotted from the tables of Dranitsina [21]), who carried out the interpolation with respect to Z, and from the energies given by Rose [7]). The upper curves were plotted from the tables of Band and Sliv [12]). The accuracy of the theoretical data given to us by Band and Sliv comprises $\approx 2\,\%$.

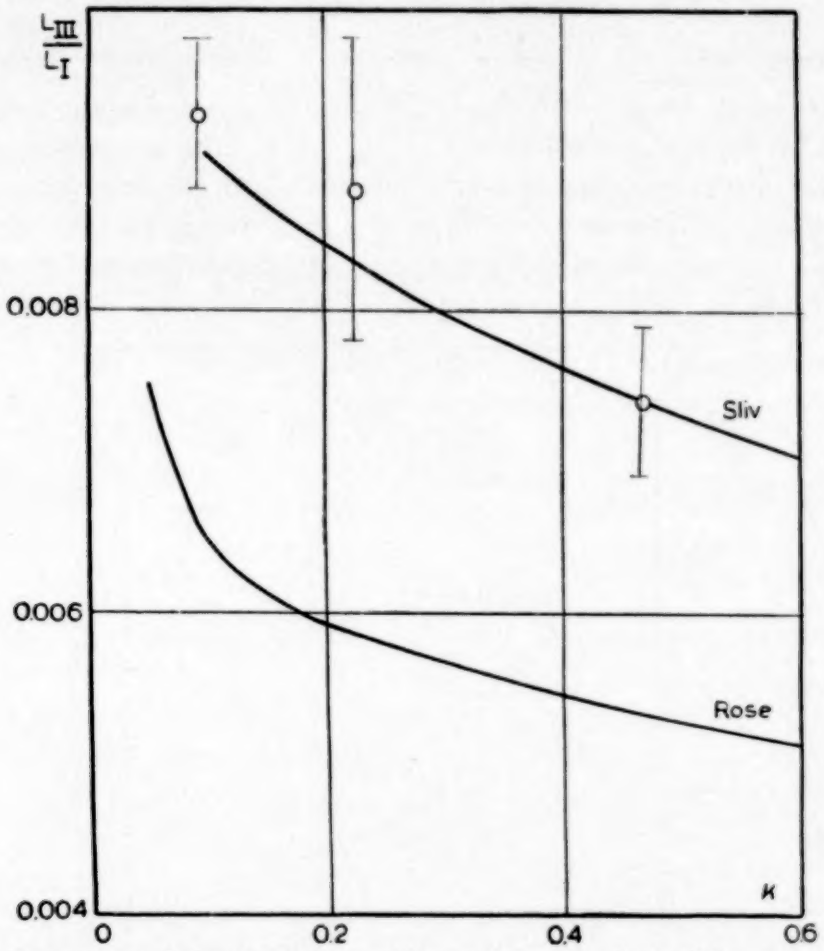

Fig. 6. Comparison of the theoretical and experimental values of the ratio $L_{III} : L_I$.

It is evident from the graphs 5 and 6 that within the limits of error the experimental data are in good agreement with the computations of Band and Sliv, in which the finite dimensions of the nucleus were taken into account. Rose's computations for a point nucleus noticeably disagree with the experiment. It thus follows from our data that the finite dimensions of the nucleus have a marked effect on the relative L conversion coefficients and that this effect has been correctly taken into consideration in the computations of Band and Sliv [12]).

As was shown in the paper [13]), in all three cases of γ-transitions studied by us a neutron goes over from the state $g_{\frac{9}{2}}$ to the state $i_{\frac{13}{2}}$, i.e., the transitions are l-forbidden. In all cases when the lifetime of the l-forbidden M1 transitions was measured a delay of 2—3 orders of magnitude was observed. All these transitions take place in odd nuclei. If there is a similar delay in the transitions in odd-odd nuclei which we investigated, then one may expect to observe the influence of the nuclear structure on the relative conversion coefficients for M1 transitions, as indicated by Church and Weneser [22]). The good agreement between our data and Sliv's computations shows that if the structure of the nucleus does influence the ratio $L_I : L_{II} : L_{III}$ in the transitions studied, then this effect is less than 5 %.

4. Relative Conversion Coefficients for the 277.3 keV γ-transition in Pb^{208}

In addition to the pure M1 transitions discussed above, we measured the ratio $L_I : L_{II} : L_{III}$ for the 277.3 keV M1 transition, which occurs upon transition between two excited levels of Pb^{208} : 3474.8 keV (4—) and 3197.5 keV (5—). In this transition there may be an admixture of E2 radiation.

For this transition we determined $L_I : L_{II}$ with a resolution of 0.05 % (fig. 7) and $L_I : L_{III}$ with a resolution of 0.23 %.

The mean values of the ratio $L_I : L_{II} : L_{III}$ for the 4—5 series of measurements are given in table 3. In addition the table gives the values $K : L_I$ as computed from the ratio $K : (L_I + L_{II})$ found in our laboratory [23]). It appears from the table that the value of $L_I : L_{II} : L_{III}$ for this transition differs markedly from the corresponding values for pure M1 transitions (compare with table 1). This difference is easily explained by the presence of a small admixture of E2 radiation in the 277.3 keV transition. Columns 4 and 5 of table 3 give the magnitude of the E2 admixture, found with the aid of Band and Sliv's new conversion coefficients[12]). It is seen that there is good agreement between the values of the E2 admixtures found from the two independent ratios $L_I : L_{II}$ and $L_I : L_{III}$. In the last two columns of the table the E2 admixture has been computed by means of Rose's coefficients [7]).

The admixture of E2 radiation in the 277.3 keV transition comprises 5 %. It is thus seen that if the accuracy of the measurements is sufficiently high, the new conversion coefficients make it possible to determine the admixture of E2 radiation even in the case when it comprises no more than several percent of the main radiation. Such small admixtures were heretofore determined mainly from angular correlation measurements.

The γ-transitions studied in the present paper belong to nuclei close to a nucleus with twice filled shells (Pb^{208}_{82}). If the probability of γ-transitions in such nuclei is correctly described by Moszkowski's formula for one-

particle transitions, then for the 277.3 keV transition the ratio E2 : M1 should be equal to $\approx 10^{-4}$. The results of our measurements show that

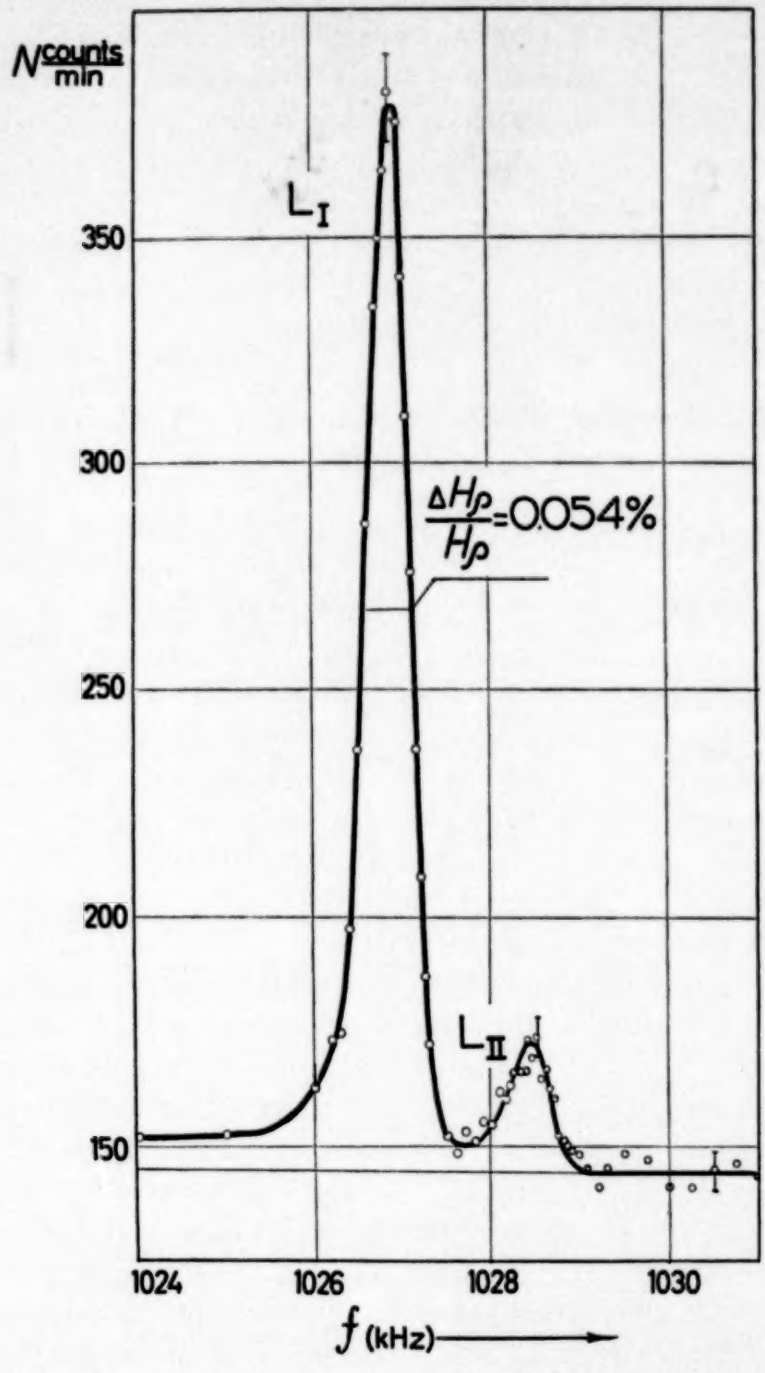

Fig. 7. L_I and L_{II} are conversion lines for the 277.3 keV transition in Pb^{208}.

this ratio is equal to 0.05. The noticeable admixture of E2 radiation may be connected with the delay of the M1 transition.

As for the 277.3 keV transition in Pb^{208}, the delay of the M1 radiation in this case cannot be explained by the structure of the excited levels of Pb^{208}

proposed in the paper [25]). According to this paper, the 277.3 keV transition takes place between the levels of one multiplet. The probability of such a transition should coincide with the one-particle estimate. This has been

TABLE 3

E_γ (keV)	K : L_I	$L_I : L_{II} : L_{III}$	Admixture of E2 in % of			
			$L_I : L_{II}$	$L_I : L_{III}$	$L_I : L_{II}$	$L_I : L_{III}$
			According to Sliv		According to Rose	
277.3	6.15±0.3	100 : (12.5±0.6) : (1.9±0.3)	4.8±1.5	5.2±1.4	11.0±1.5	8.8±1.8

confirmed experimentally for the case of the 40 keV M1 transition in Tl^{208}. This transition occurs between the levels of the multiplet $s_{\frac{1}{2}}\ g_{\frac{9}{2}}$, and direct experimental measurements of the lifetime, ref.[26]), are in agreement with shell model computations (ref.[27])). In addition, the ratio $L_I : L_{II} : L_{III}$ [28]) for this transition excludes an admixture of E2 radiation exceeding 0.05 %. It is also difficult to expect a noticeable delay of the M1 transition for transitions between the levels of any other multiplet. Hence, the presence of a noticeable admixture of E2 radiation in the 277.3 keV transition in Pb^{208} is in contradiction to the assumption (ref.[25])) that the initial and final levels of this transition belong to the same multiplet.

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RESONANCE ENERGIES AND HALF-WIDTHS FOR PROTON CAPTURE IN ALUMINIUM

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Abstract: A description is given of the methods used to obtain precise values for the resonance energies and half-widths in proton capture reactions. A target chamber designed for this purpose is described. Proton capture in Al^{27} has been investigated in the proton energy range 500 keV to 1400 keV. For all known resonances energy values are given. For some of the resonances the true half-widths and for the rest upper half-width limits are given.

1. Introduction

The width of a resonance, which is at the same time a measure of the lifetime of the excited state in question, is in itself a nuclear quantity of great theoretical interest. These widths decrease fast with increasing atomic weight, and for nuclei with $A > 20$ very good energy resolution is required if they are to be measured. In addition resonances sometimes are so closely spaced that good resolution is required to resolve them. For these reasons we have tried to measure resonance widths as carefully as possible. A study has been undertaken of some of the difficulties encountered in such experiments, and we have developed a method of observation by which many of the usual sources of error are eliminated. Some of our results have been given earlier [1,2]). In this paper we give an account of the experimental technique and its application on width measurements in the reaction Al^{27} (p, γ) Si^{28}.

2. Experimental Equipment

2.1. PROTON BEAM

Protons from the Blindern Van de Graaff accelerator were analyzed in a 90° electrostatic analyzer of the type described by Warren *et al.*[3]) and discussed by Herb *et al.*[4]).

Mean radius of curvature of our analyzer is 1000 mm. The entrance and exit slits are placed respectively 200 mm and 530 mm from the ends of the deflecting plates, giving a magnification M of the analyzer of about -1.2.

The analyzer is operated with slit widths from 0.2 mm and upwards, depending on the degree of resolution required. Although the magnification is somewhat larger than 1, the same width setting is used in entrance and exit slit.

If the beam density is constant over the width of the entrance slit and the number of protons per unit energy range is constant over the range of energies accepted by the analyzer, the proton energy distribution after passing the exit slit will have the form of a symmetrical truncated triangle. The total energy spread of the analyzed protons, neglecting instability in the analyzer plate voltage, is given by the formula $\Delta E = 2[(s-Ms')/a(1-M)]\,E$, where s' and s are the entrance and exit slit widths respectively and a the mean radius of curvature. The energy range over which the protons are equally distributed (the shorter side of the truncated triangle) is $2[|s+Ms'|/a(1-M)]\,E$. From these expressions the inter-quartile range of the energy distribution may be calculated. For $s = s'$ and $M = -1.2$ we find the inter-quartile range to be $0.6\ (s/a)\ E$.

The analyzer plate voltage is stabilized to better than 0.01 % and voltage differences of about 0.005 % can be measured on a relative scale.

2.2. TARGET TECHNIQUE

As pointed out in earlier papers [1,2]), extremely great care has to be taken in preparation and use of targets. When the utmost precision is needed, aluminium targets must not be exposed to air after they have been produced. Therefore they have to be evaporated in the target chamber. Due to carbon deposit and perhaps other effects, bombardment of the same spot on the target causes shifts in position and shape of a resonance curve after a few minutes, even when a liquid air trap is placed near the target. In our experience heating of target to about 300° C does not seem to give satisfactory results when a resolution of 200 eV is required. Therefore a multitude of bombarding spots must be available on the same target.

A target chamber fulfilling these requirements was made for these experiments. A drawing of the chamber is shown in fig. 1.

The target backing is a cylinder of copper or tantalum (4). The target film is applied by evaporation of the target material from the filament in the evaporation chamber (7) while the target is continuously rotated by turning quickly the shaft of the tooth wheel (11). In this way a uniform broad layer of the target material is obtained on the cylinder. When being used, the target may be turned in steps, defined by the tooth distance of the wheel (11), giving one hundred different target spots on the circumference of the target.

Furthermore the cylinder can be displaced along its axis, in positions defined by the locking pin (2) and the grooves (10) in the outer target shaft

so that four different rows are obtained, giving in all 400 fresh target spots from one evaporation.

For gamma ray measurements room is provided for a scintillation counter in the well (6) inside the target cylinder, situated as close to the target as possible to give the highest sensitivity for detection.

The target chamber is also intended to be used for counting positons from reactions like $Mg^{25}(p, \gamma)Al^{26}$, and for this purpose the cylinder (4) may be turned 90° by the handle (1), bringing the bombarded spot just under a Geiger Müller tube placed in the opening (3). The handle (1) has

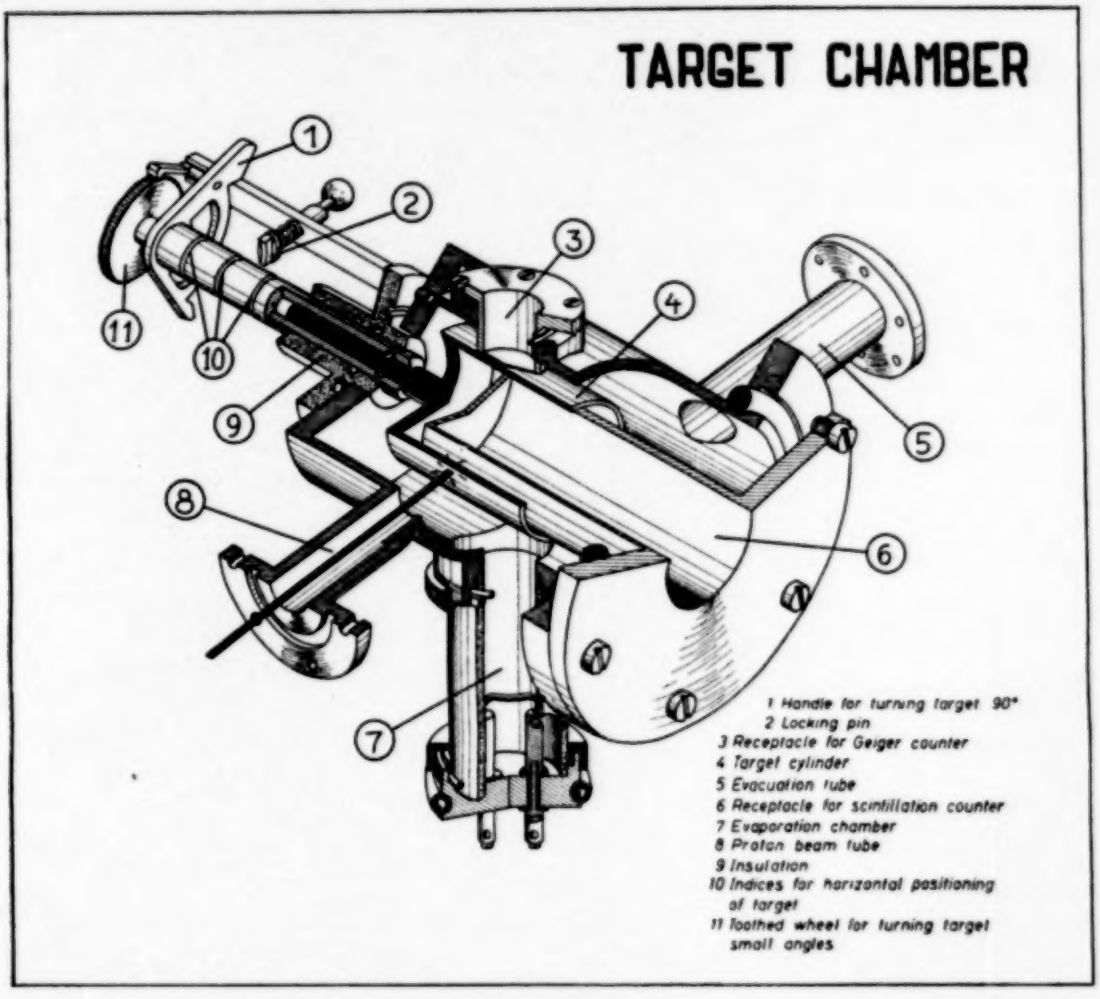

Fig. 1. Target chamber.

two defined end positions and turns the target by an outer shaft, so that the continuous sequence of bombarding spots defined by the position of the tooth wheel (11) is not disturbed by turning the outer shaft for positon detection.

All resonances reported here, except for the broadest one, were measured on targets being thick compared to the resonance width, thus giving an integrated yield curve. In our experience this is the only satisfactory way to

obtain a correct value for the width of a narrow resonance. Targets thinner than 1 keV stopping power consistently seem to have non-uniform thickness, and also the yield decreases for targets thinner than either the resonance width or the proton beam energy spread. Whenever possible, efforts were made, on the other hand, to keep target thin enough to exclude proton capture in the next lower resonance.

3. Experimental Results

3.1. PROTON ENERGY MEASUREMENTS

The most commonly used energy scale in recent years is based on Herb's [4]) value 993.9±1.0 keV for the strong resonance in the reaction $Al^{27}(p, \gamma)Si^{28}$.

TABLE 1

Resonance energies and half-widths for all resonances reported between 500 keV and 1400 keV.

No.	Resonance energy (keV)	Γ_{obs}(eV)	$(\Sigma\Gamma_i^2)^{\frac{1}{2}}$(eV)	Γ(eV)
1	504.5±0.3	180±50	120	< 200
2	506.5±0.3	160±50	120	< 170
3	612.4±1			< 1000*
4	632.3±0.3	100±40	140	< 60
5	654.3±0.3	100±40	140	< 60
6	677.6±1			< 1000*
7	730.6±0.3	180±40	150	< 160
8	735.6±0.3	130±40	150	< 90
9	741.1±0.7			< 1000*
10	759.4±0.3	120±40	160	< 60
11	766.3±0.3	130±40	160	< 80
12	772.8±0.3	180±50	160	< 160
13	884 ±2*			< 1000*
14	921.7±0.3	210±50	180	< 190
15	936.3±0.3	390±100	190	340±110
16	990.8	220±40	200	< 170
17	1001.5±0.5			< 1000*
18	1023.9±0.3	260±50	200	< 240
19	1088.6±0.3	180±50	210	< 110
20	1096.4±0.5			< 1000*
21	1117.2±0.3	830±80	220	800±80
22	1170.6±0.4	290±50	230	< 250
23	1182.0±0.4	740±130	230	710±130
24	1198.0±0.6	6400±400		6300±400
25	1211.6±0.4	260±50	230	< 210
26	1260.7±0.4	260±50	240	< 200
27	1274 ±2*			< 1000*
28	1315.2±0.4	230±50	250	< 160
29	1326.6±0.4	230±50	250	< 160
30	1362.0±0.4	210±50	260	< 120
31	1379.6±0.4	750±50	260	700±50
32	1386.7±0.4	390±50	260	290±70

The column for $(\Sigma\Gamma_i^2)^{\frac{1}{2}}$ indicates the value to be expected for Γ_{obs} if Γ is small. For the sake of completeness values are included for the weakest resonances taken from ref.[4]) and adjusted to our present energy scale. These are marked with an asterisk.

A more recent value, claiming greater accuracy, is 990.8 ± 0.2 keV for the same resonance, obtained by Bumiller *et al.*[5]). We have accepted the latter as our standard and have measured the energies of all the resonances known in the region of 500—1400 keV except for the weakest ones. The results are given in table 1.

The uncertainty in reproducing a resonance value in our scale is 0.02 %, the same order of magnitude as the limits of error given by Bumiller for the calibration energy.

As reported earlier [2]) we find a difference in energies obtained by measuring the same resonance with the H^+ and H_2^+ beam. As long as the reason for this discrepancy is not understood, we find it most consistent to assume the scale to be linear and give the values obtained with the H^+ beam.

If the discrepancy is due to non-linearity in energy scale, this means that a systematic error is included which, when corrected for, would reduce the values of resonances situated below 991 keV and raise the values of those situated above this energy. In the region around 500 keV the values given would be about 0.08 % too high, and around 1250 keV about 0.06 % too low. This uncertainty is not included in the errors given in table 1.

For the broad resonance at 1198 keV, the resonance energy given is the proton energy corresponding to the peak of the resonance minus half of the target thickness. All the others are integrated yield curves, and we interpret the proton energy corresponding to one half of maximum yield as the resonance energy. Comparing the results with those of Broström, Huus and Tangen [6]), we find very good agreement. The only significant deviation is the change in energy scale. Using a value of 503 keV [7]) for the lowest of the reported resonances, they obtained 986 keV for the strong resonance for which we used, as calibration point, the value 990.8 keV. As a consequence our results generally come out 4—5 % higher.

3.2. RESONANCE WIDTHS

The form of a resonance is most commonly described by the Breit-Wigner formula with the parameter Γ, usually called the resonance half-width. Γ also equals the interquartile range on a distribution curve following the Breit-Wigner equation.

Experimentally the half-width of a distribution is directly measured on a differential curve. On an integral curve, however, the interquartile range is a quantity which can be measured directly, whereas the half-width can be calculated when the form of the curve is known. Only for special distributions, as the Breit-Wigner distribution, the two quantities are equal. For a narrow resonance the yield curve does no longer rise according to the Breit-Wigner formula, but depends also on energy spread in the proton beam, and on thermal motion of the target molecules. As we do our measurements on

thick targets, we obtain integral curves and find the interquartile ranges Γ_{obs} which are to be compared with the Breit-Wigner half-width Γ. From Γ_{obs} we obtain a corrected value of Γ by using the approximation $\Gamma_{obs}^2 = \Gamma^2 + \sum \Gamma_i^2$ where the Γ_i's are estimated interquartile ranges for the energy spreads due to the various effects. The main effects are the following:

a) Energy spread of proton beam due to finite slit width. The corresponding interquartile range is $\Gamma_a = 0.6(s/r)E$. The setting of the slit width is believed to be correct within 10 %.

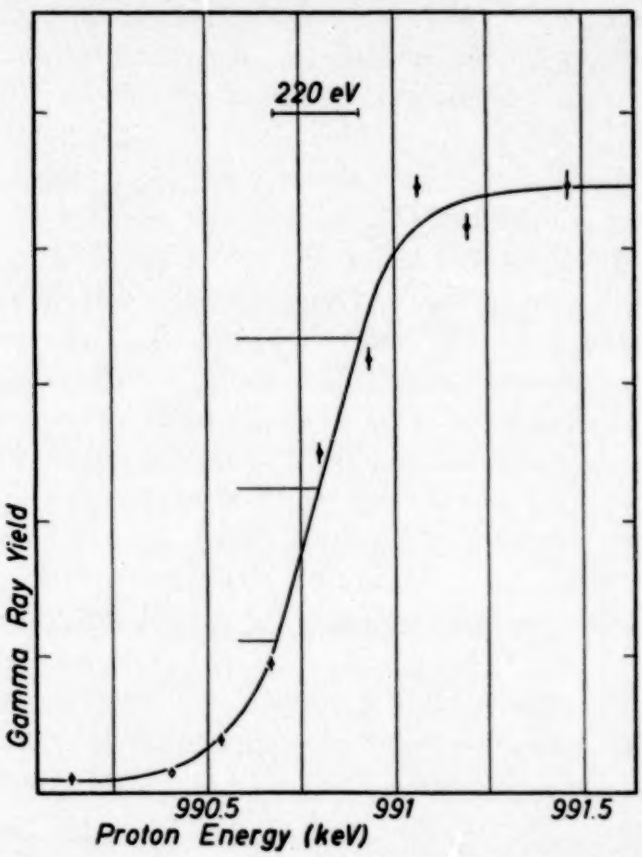

Fig. 2. The resonance at 990.8 keV measured with slit width 0.3 mm.

b) Energy variation in proton beam due to instability of the analyzer plate voltage and uncertainty in the voltage reading. This is observed to be less than ± 50 eV in the energy range used. The corresponding interquartile range of the energy spread is estimated to be half of this, i.e. $\Gamma_b = 50$ eV.

c) Doppler broadening of the resonance due to thermal motion of the target nuclei. According to Lamb [8]) Doppler broadening gives a standard deviation of an infinitely sharp resonance $\Delta = (2E_p k\bar{\varepsilon} m/M)^{\frac{1}{2}}$ where m is the proton mass, M the mass of the molecule, E_p the proton energy, k the Boltzmann constant and $\bar{\varepsilon}$ the effective temperature of the target molecules. In our case the target is kept at room temperature. There may be a local heating of the bombarded target spot, but we assume it to be small and set the

effective temperature of the aluminium target $\bar{\varepsilon} = 360°$ corresponding to a target temperature of 300° K.

Inserting the proper values we find $\Delta = 50\sqrt{E_p}$ eV where E_p is measured in MeV. The corresponding interquartile range of a normal distribution function $\Gamma_D = 1.36\Delta \approx 70\sqrt{E_p}$ eV.

Bumiller *et al.*[9]) have measured the half-width of the strong resonance at 990.8 keV and found it to be 50 ± 30 eV. We have used the same resonance to check our results. In fig. 2 is given a yield curve measured with slit width of 0.3 mm.

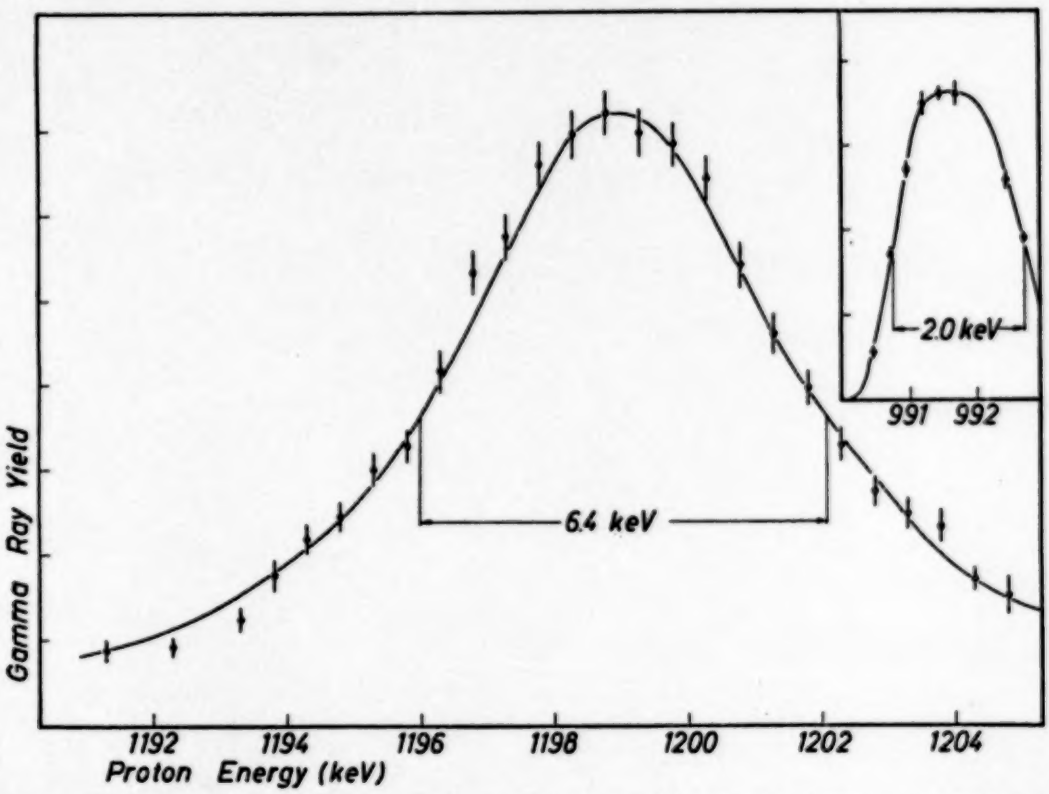

Fig. 3. The resonance at 1198 keV. The curve is calculated by the Breit-Wigner formula for a semi-thick target[15]) using a resonance energy of 1198 keV, a half-width of 6.3 keV and a target thickness of 1.8 keV. The insert shows the 990.8 keV resonance measured on the same target.

Given as interquartile ranges, the corrections to be made are:
from slit width setting: $\Gamma_a = 0.6(s/r)E = 180\text{ eV}\pm20\text{ eV}$,
from instability: $\Gamma_b = 50$ eV,
from Doppler broadening: $\Gamma_D = 70$ eV.

Adding these contributions to the known half-width of 50 ± 30 eV, we get an expected value for the interquartile range for the yield curve $\Gamma_{obs} = (\Gamma_a^2+\Gamma_b^2+\Gamma_D^2+\Gamma^2)^{\frac{1}{2}} = 205\text{ eV}\pm25\text{ eV}$. The observed value is 220 ± 40 eV.

We have measured the widths of all but the very weakest of the resonances known to exist between the proton energies of 500 keV and 1400 keV. Except for the resonance at 991 keV, the best measurements in this region

were those by Broström, Huus and Tangen [6]). They found a half-width of less than 1 keV for all of the resonances except for one at 1198 keV (on our

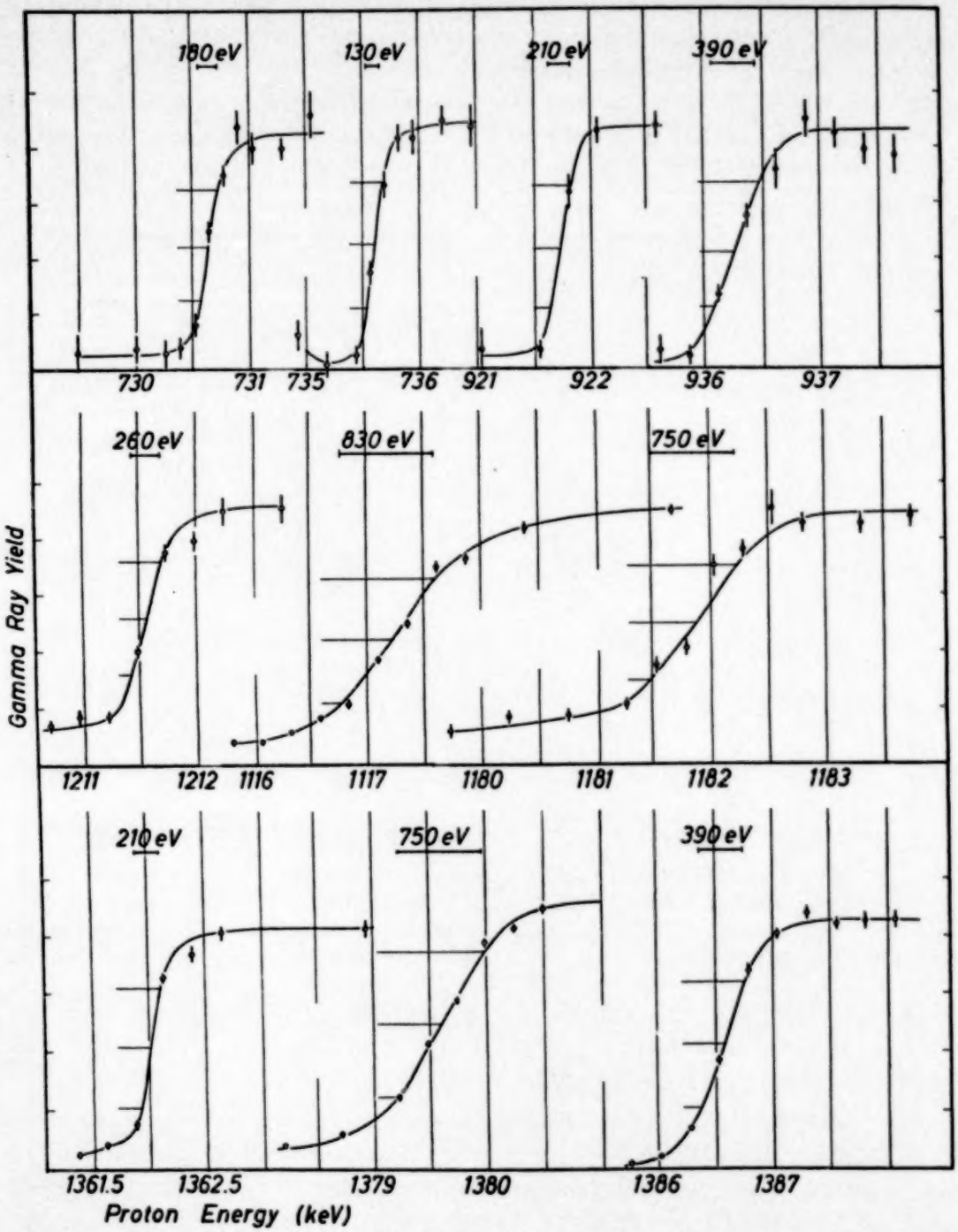

Fig. 4. Thick target gamma ray yield curves for 10 resonances, including all those with a measurable width (except the one at 1198 keV) with some narrow resonances added for comparison. The ordinate is adjusted to give the same height for all resonances. For each one is indicated the interquartile range Γ_{obs}.

scale) which seemed to be an unresolved group of three resonances. With the present higher resolution we have tried to resolve the assumed group, using a variety of target thicknesses to bring out the best results, and conclude that there is only one broad resonance at 1198 keV (fig. 3). As an extra check we measured the gamma ray spectra at 1196, 1198 and 1200 keV. The spectra turned out to be identical, indicating that only one level of the compound nucleus is involved.

For the resonance found by Tangen [7]) at 503 keV, Hunt and Jones [10]) found a half-width of (0.7 ± 0.3) keV. As reported earlier we have shown that it consists of 2 narrow resonances with a spacing of 2 keV.

For 5 other resonances in the energy region above 900 keV, we find half-widths of the order of 0.5 keV. For the rest of the resonances we get lower limits of uncertainty which are zero or below, and for these resonances we only give upper limits for the half-widths.

In table 1 is given for each resonance the calculated value of $(\sum\Gamma_i^2)^{\frac{1}{2}}$ together with Γ_{obs} from which is found a value for Γ.

4. Discussion

According to generally accepted nuclear resonance theory the total half-width of a resonance is the sum of the widths (or rates of disintegration) of all the channels through which the corresponding level in the compound nucleus decays. The energetically possible modes of decay in our cases are alpha emission, gamma transitions and proton re-emission with the contributions to the total half-width Γ_α, Γ_γ and Γ_p respectively.

Yield measurements by Shoemaker *et al.*[11]) show the α-yield to be smaller than the resonant proton yield and we therefore assume Γ_α to be smaller than Γ_p.

A compilation of experimental values of transition probabilities by Wilkinson [12]) indicates a value for Γ_γ of 60 eV, with the statistical spread within a factor of 7 either way for an electric dipole transition of 12.5 MeV energy. However, most of the states in question decay through intermediate levels, which means that the transition to the ground state is forbidden, and that Γ_γ must be smaller, in agreement with the conclusion arrived at by Broström, Huus and Tangen [6]), that Γ_γ is of the order of 1 eV. Our gamma ray spectra show that the broad resonance at 1198 keV decays to the second or higher excited states, so Γ_γ cannot be very large in this case.

Blatt and Weisskopf [13]) give arguments that the proton width, Γ_p, for S wave protons rises from the order of 1 eV around 600 keV to about 200 eV around 1400 keV. They assume most of the levels to be due to S wave protons. The results of Rutherglen *et al.*[14]) indicate P wave capture to be just as frequent, which tends to increase the value for the level distance, and thereby the estimate for Γ_p, by a factor of 2 or 3.

This gives fairly good agreement between the theoretical estimate of the width for S wave protons, and half-widths measured in this work. The radiation width and the alpha particle width are probably too small to contribute appreciably to the measured half-widths.

5. Conclusion

Although this work was done with higher energy resolution and improved detector efficiency as compared with earlier works, we have found only one new resonance, by splitting the one formerly known at 503 keV. We have been able to measure the true resonance width of 6 of the resonances, most of them in the upper half of the region covered.

Since the broad resonances are not generally α-emitting, and since such large radiation widths are improbable, the measured widths most probably are the proton widths.

The authors wish to express their gratitude to Mr. J. Sandstad for his very valuable help with the electronic equipment.

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SHAPE FACTORS FOR FIRST FORBIDDEN NON-UNIQUE BETA TRANSITIONS

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Abstract: New approximations are developed for the shape factors in non-unique beta decay. Simplifications are possible because the combination of the so-called forbidden beta-decay functions $(L_p M_p)^{\frac{1}{2}} \pm N_p$ (upper sign for negatons, lower sign for positons) is very small as compared to the functions themselves.

1. Introduction

The theoretical formulae for the shape of continuous beta spectra as given by the Fermi theory [1]) are rather unwieldy, especially in the case of non-unique transitions where the shape is determined by the accidental values of the nuclear matrix elements. This is even true if the beta-decay interaction contains only S, T and P terms, as assumed until recently [1, 2]), or only V and A interactions as would follow from the latest experiments [19, 20]) as well as from the theory of Feynman and Gell-Mann [3]).

A knowledge of the spectral shapes is important in measurements of internal conversion coefficients of gamma rays by the method of comparing internal conversion line intensities with the intensity of a continuous beta spectrum. Many measured beta spectra are significantly distorted for momenta less than about half the maximum momentum on account of back-scattering in the source and in the source backing and also by scattering in different parts of the beta-ray spectrometer. The lower part of the spectrum has therefore to be constructed by extrapolation from the higher energy measurements, using the theoretical spectral shape. The spectral shape should be known with a precision of a few percent, if the conversion coefficients are to be determined with a slightly higher accuracy. We have therefore tried to find a useful approximation for the spectral shapes in non-unique transitions.

2. Results of Beta Decay Theory

The theoretical energy distribution in an n^{th} forbidden electron spectrum is

$$N(p)\mathrm{d}p = (g^2/2\pi^3)p^2q^2F_0S_n/(n!)^2\mathrm{d}p; \qquad (2.1)$$

p and q are the electron and neutrino momenta and F_0 the Fermi function. Formulae for the shape factor S_n are given by Greuling [4]) and Pursey [5]) and (more correct for pseudoscalar interaction [8])) by Konopinski and Uhlenbeck [6]) and Smith [7]). Rearranging their terms we can write

$$S_n = S_{n,n} + S_{n,n+1} + \delta_{n,1} S_{10} \tag{2.2}$$

where, for a combination of vector and axial vector interactions,

$$S_{n,n} = \sum_{\nu=0}^{n-1} (n-\nu)(2n-2\nu+1) B_{n\nu} q^{2(n-\nu-1)} \Big[M_\nu (v_n \pm a_n)^2 + 2N_\nu (v_n \pm a_n) \Big(\xi v_{n\mathrm{a}} - \frac{v_n \mp a_n}{2n-2\nu+1} q \Big) + L_\nu \Big(\xi v_{n\mathrm{a}} - \frac{v_n \mp a_n}{2n-2\nu+1} q \Big)^2 \Big], \tag{2.3}$$

$$S_{n,n+1} = \sum_{\nu=0}^{n} B_{n\nu} q^{2(n-\nu)} L_\nu \left[\frac{a_{nu}^2}{(n+1)^2} + \frac{(n+1)}{(2\nu+1)(2n-2\nu+1)} \left(v_n \pm \frac{n-2\nu}{n+1} a_n \right)^2 \right],$$

$$S_{10} = M_0 a_{10}^2 - 2N_0 a_{10} (\xi a_{\mathrm{a}1} + \tfrac{1}{3} a_{10} q) + L_0 (\xi a_{\mathrm{a}1} + \tfrac{1}{3} a_{10} q)^2.$$

The upper signs refer to negatons, the lower ones to positons The parameter ξ is defined

$$\xi = \hbar \alpha Z / 2mcr \approx 0.42 Z A^{-\frac{1}{3}}. \tag{2.4}$$

In most cases ξ is a rather large number: $\xi \approx \pm 10$. The Greuling constant $B_{n\nu}$ can be written in the following symmetric way:

$$B_{n\nu} = \frac{1}{\nu!(n-\nu)!} \frac{(2\nu+1)!!}{(2n-2\nu+1)!!}. \tag{2.5}$$

The parameters v and a are combinations of interaction constants and matrix elements explained in a symbolic way in table 1. In a general parity non-conserving theory, their squares and products should be defined, e.g.,

$$v_0^2 = (|C_{\mathrm{V}}|^2 + |C'_{\mathrm{V}}|^2) \Big| \int 1 \Big|^2,$$

$$2 v_0 a_{0\mathrm{u}} = \mathrm{Re}\, (C_{\mathrm{V}} C_{\mathrm{A}} + C'_{\mathrm{V}} C'_{\mathrm{A}}) \Big[\int 1 \int \boldsymbol{\sigma}^* + \mathrm{c.c.} \Big].$$

Then the relation $(v_0 a_{0\mathrm{u}})^2 = v_0^2 a_{0\mathrm{a}}^2$ generally does not hold so that the formulae (2.3) have to be used with some caution. The normal relations between squares and products are, however, valid in the two-component neutrino theory with time reversal and in the Feynman–Gell-Mann theory.

Formulae for a combination of scalar, tensor and pseudoscalar interactions are obtained by the following substitutions:

$$q \to -q, \qquad v_n \to s_n, \qquad a_n \to t_n, \qquad v_{n\mathrm{a}} \to \pm t_{n\mathrm{a}},$$
$$a_{n\mathrm{u}} \to t_{n\mathrm{u}}, \qquad a_{10} \to t_{10}, \qquad a_{\mathrm{a}1} \to \pm p_{\mathrm{a}1}.$$

The parameters s, t and p (column 2 in table 1) are defined in the same way as the corresponding v and a's in the first column except that a factor β has to be added under the integral signs.

TABLE 1

Definition of parameters in formula (2.3)

Our notation		Usual Notation			Selection Rules		
VA	(STP)	$n=0$	$n=1$	$n=2$	J	J_i+J_f	$\pi_i\pi_f$
v_n	(s_n)	$C_V\int 1$	$C_V\int i\boldsymbol{r}$	$C_V R_{i_1\dots i_n}$	$\leqq n$	$\geqq n$	$(-1)^n$
a_n	(t_n)		$C_A\int \boldsymbol{\sigma}\times\boldsymbol{r}$	$C_A T_{i_1\dots i_n}$	$\leqq n$	$\geqq n$	$(-1)^n$
ξv_{n3}	(ξt_{n3})		$C_V\int \boldsymbol{\alpha}$	$C_A A_{i_1\dots i_n}$	$\leqq n$	$\geqq n$	$(-1)^n$
a_{n3}	(t_{n3})	$C_A\int\boldsymbol{\sigma}$	$C_A B_{ij}$	$C_A B_{i_1\dots i_{n+1}}$	$\leqq n+1$	$\geqq n+1$	$(-1)^n$
a_{10}	(t_{10})		$C_A\int\frac{1}{i}\boldsymbol{\sigma}\cdot\boldsymbol{r}$		0		-1
ξa_{31}	(ξp_{31})		$C_A\int\gamma_5$		0		-1

A combination of all five interactions would, in addition, give rise to Fierz terms containing combinations of matrix elements $\mathrm{Re}(C_{VA}C_{STP}+C'_{VA}C'_{STP})$. The results of measurements of longitudinal polarization of electrons [9]) indicate that (if VA and STP interactions occur in comparable amounts) $C'_{VA}\approx+C_{VA}$ and $C'_{STP}\approx-C_{STP}$ (in agreement with the prediction of Feynman and Gell-Mann [7])); the Fierz terms should therefore be small. Spectrum shape measurements [10]) point to the same conclusion; a similar and even more accurate result is obtained from measurements of electron capture to positon emission ratios on ^{22}Na [11]), ^{18}F [12]), ^{48}V and ^{57}Ni [13]). We shall therefore disregard these terms.

Transition rates for electron capture in the X_σ shell can be computed from the combination of eqs. (2.1—3) if the functions L_ν, N_ν and M_ν, multiplied by $2p^2F_0$, are replaced by functions defined as follows:

X_{2n-1} shell: $L_{n-1}=n\pi R^{2-2n}f^2$ $\quad$ X_{2n} shell: $L_{n-1}=n\pi R^{-2n}g^2$

$N_{n-1}=n\pi R^{1-2n}fg$ $\quad$ $N_{n-1}=n\pi R^{-1-2n}fg$

$M_{n-1}=n\pi R^{-2n}g^2$; $\quad$ $M_{n-1}=n\pi R^{-2-2n}f^2$.

L_ν, N_ν and M_ν vanish for $\nu\neq n-1$; R is short for $mcr/\hbar$, r being the radius of the nucleus; on the right hand sides, the factor n has to be replaced by half the number of electrons in the shell under consideration, if this shell is not completely filled. Data about the functions f^2 and g^2 in the K and L shells can be found in a report by Brysk and Rose [14]).

3. The New Approximation

The functions L, M and N have the following properties:

$$M_\nu\approx L_\nu\xi^2/(\nu+1)^2, \tag{3.1}$$

$$N_\nu\approx\mp(L_\nu M_\nu)^{\frac{1}{2}}\approx-\xi L_\nu/(\nu+1). \tag{3.2}$$

The upper sign again applies to negaton emission and electron capture, the lower to positon emission. Eq. (3.2) is exact for electron capture, and sur-

prisingly accurate for electron emission in all practically occurring cases, as is proved as follows. We define

$$O_\nu = R^{-1-2\nu}(f_{\nu+1}f_{-\nu-1}-g_{\nu+1}g_{-\nu-1}).$$

Then $L_\nu M_\nu = N_\nu^2+O_\nu^2$. Using second order approximations for the electron wave functions we find

$$O_\nu = p/(2\nu+3)W. \tag{3.2'}$$

Taking all these data together, we derive

$$\frac{(L_\nu M_\nu)^{\frac{1}{2}}\pm N_\nu}{N_\nu} = \frac{(\nu+1)^2}{2\xi^2(2\nu+3)^2}\left(\frac{P}{W}\right)^2 \lesssim 0.02\ \%.$$

In fact we found from the tables by Rose *et al.*[15]) that the difference between $\mp N_\nu$ and $(L_\nu M_\nu)^{\frac{1}{2}}$ is at most 0.3 % for negatons and $Z = 10$, and considerably smaller for larger Z. For positons a complication arises due to the fact that N_ν has a zero (and M_ν a minimum). This zero occurs, however, for large energies (5 MeV for $Z = 10$, 8 MeV for $Z = 20$, and still higher energies for higher Z). For all known first forbidden positon transitions the error introduced by replacing $\mp N_\nu$ by $(L_\nu M_\nu)^{\frac{1}{2}}$ would again be smaller than 0.3 %.

Combining eqs. (2.2) and (3.2) one obtains an expression for the part of $S_{n,n}$ between parentheses which is nearly a square. $S_{n,n}$ can therefore be written

$$\begin{aligned} S_{n,n} &= \sum_{\nu=0}^{n-1}(n-\nu)(2n-2\nu+1)B_{n\nu}\xi^2 q^{2(n-\nu-1)}L_\nu(C_{n\nu}+b_{n\nu}q)^2+P_{n\nu}, \\ C_{n\nu} &= v_{n\mathfrak{a}}-(v_n\pm a_n)\frac{1}{\xi}\left(\frac{M_\nu}{L_\nu}\right)^{\frac{1}{2}}, \qquad b_{n\nu} = \frac{1}{\xi}\frac{v_n\mp a_n}{2n-2\nu+1}. \end{aligned} \tag{3.3}$$

The quantity S_{10} can be simplified in a similar way. The quantities $P_{n\nu}$, and a fortiori their influence on the energy dependence of the correction factors S_n, are small compared with the other terms except when

$$v_{n\mathfrak{a}}(v_n\pm a_n)/C_{n\nu}^2 \gg 1. \tag{3.4}$$

This can only arise as a result of an accidental cancellation of matrix elements, which is in general improbable. A similar relation holds for S_{10}.

4. First Forbidden Transitions

We consider first forbidden transitions with spin change and with normal *ft* values, meaning that no accidental cancellation of matrix elements occurs. Then, if the decay energy is not too high, only the shape factor $S_{1,1}$ is important; and in eq. (3.3) we may take $P_{1,0} = 0$. The parameter $(M_0/L_0)^{\frac{1}{2}}$ in this formula is a nearly linear function of the electron energy (compare fig. 1) and if we write

$$(M_0/L_0)^{\frac{1}{2}}/\xi = \pm d+eq/\xi, \tag{4.1}$$

then $d \approx 1$ and $e \approx 0.5$. The function L_0 is also a nearly linear function of the electron energy (compare fig. 2):

$$L_0 \propto (1-fq/\xi); \tag{4.2}$$

f is small for low Z and approaches 0.5 for high Z. Combining these approximations with eq. (3.3) we obtain

$$S \propto (1+gq)^2, \tag{4.3}$$

where

$$g = \frac{1}{3\xi}\left[\frac{v_n(1-3e) \mp a_n(1+3e)}{v_{na}-d(a_n \pm v_n)} - \tfrac{3}{2}f\right]. \tag{4.4}$$

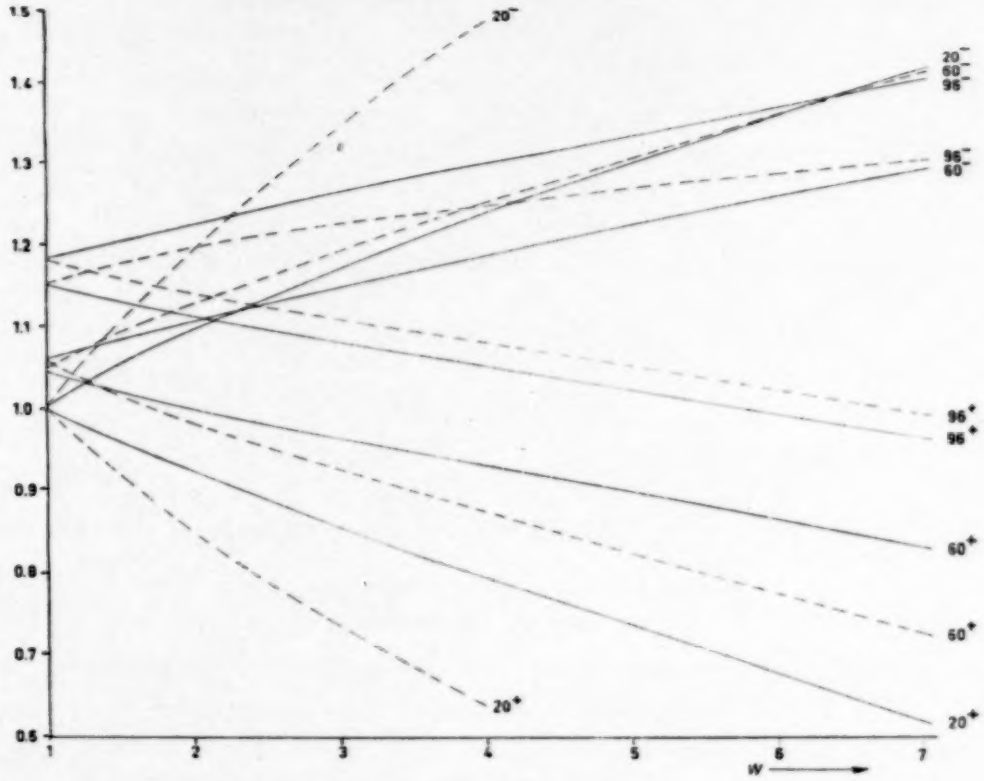

Fig. 1. The functions M_0/ξ^2(dotted) and $(M_0/L_0\xi^2)^{\frac{1}{2}}$ (drawn) as a function of the relativistic electron energy. The Z values are indicated; the attached signs refer to positons (+) or negatons (−).

The term S_{10} which should be added for transitions without spin change can be written in a similar way; the two terms can in most cases be combined without significant loss of precision into one shape factor having the same form as eq. (4.3).

In view of the above considerations we propose that first forbidden transitions should be analysed by plotting $(N/p^2F_0)^{\frac{1}{2}}/W_0-W)$ as a function

of the electron energy; this plot should yield a straight line with a slope g. The best value for W_0, being that value for which the plot is most nearly linear, can be obtained most easily by trial and error. (In allowed transition the same plot should give a straight line with a slope $g = f/2\xi$.)

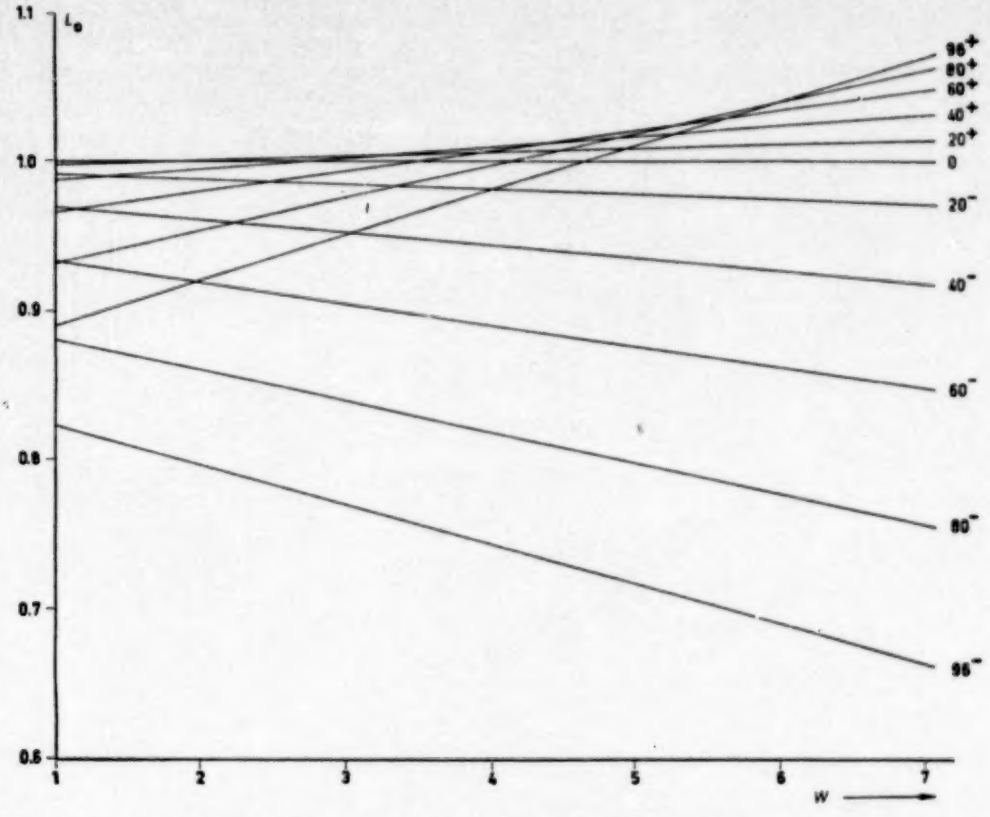

Fig. 2. L_0 for positons (+) and negatons (−) as a function of the electron energy for different values of Z. As in fig. 1, the function L_0 has been taken from the work of Rose *et al.* [15]).

5. Some Examples

We shall show the results of such an analysis on a measurement of Au^{198}. This isotope has recently been remeasured in this laboratory [16]). Fig. 3 shows the Fermi plot of one of these measurements. The deviations from a straight line are very small, in agreement with earlier observations mostly using at least ten times thicker sources; they are, however, definitely systematic. As a result we get a "forbidden plot" (fig. 4) which can be represented by a straight line above 200 keV. Fig. 5 shows the influence of this method of extrapolating the β-spectrum; the area under the curve extrapolated in the way discussed above is 2.3 % larger than that obtained in the usual way. This difference could not be neglected, since it was intended to use this area for a determination of a conversion coefficient with an error of at most a few percent. The deviations below 200 keV are probably due to scattering, as will be discussed elsewhere.

In the case of ^{198}Au the Fermi plot was found to be curved away from the energy axis, meaning that the constant g in eq. (4.6) was positive (+0.058). Porter *et al.*[7]), studying the component in the 940 keV β-spectrum of ^{186}Re

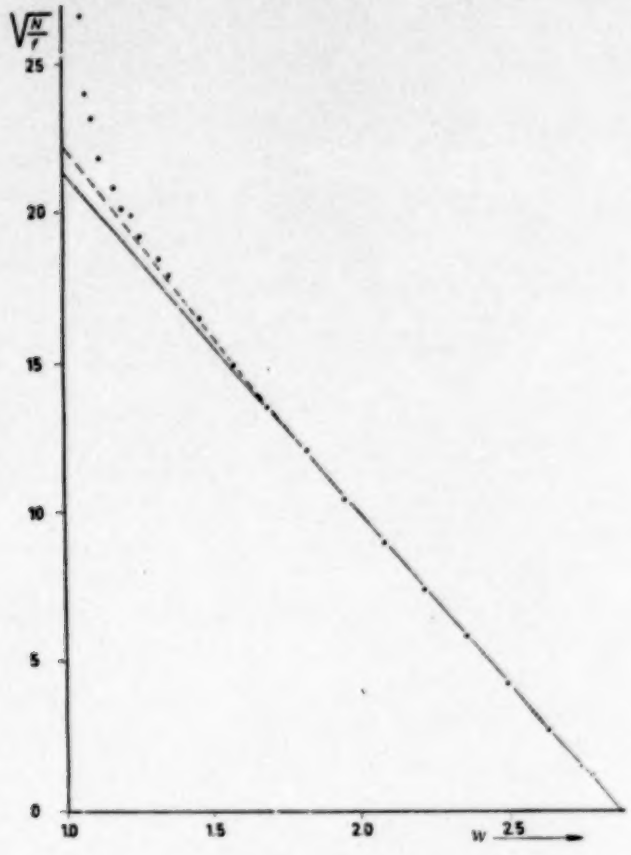

Fig. 3. Fermi plot of a measurement of the main component in the continuous beta spectrum of ^{198}Au. The dotted line is recomputed from the "forbidden plot" (fig. 4). The function f, taken from Nat. Bur. Standards, Applied Mathematics, Series 13, is proportional to p^2F as defined in section 1; it has been corrected for screening.

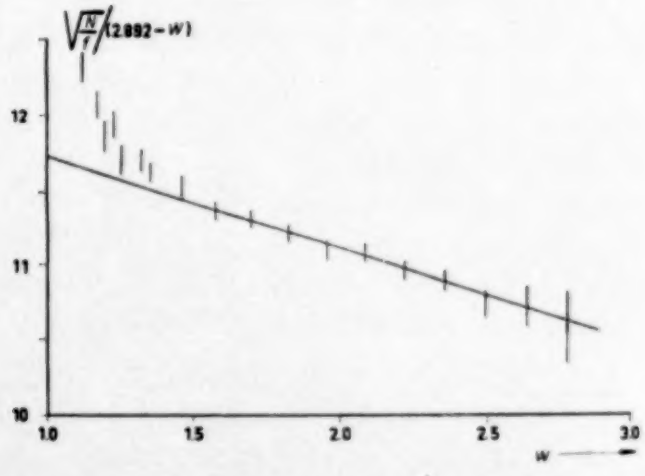

Fig. 4. First "forbidden plot" of the data given in fig. 3.

leading to the first excited state in ^{186}Os, found here a Fermi plot curved towards the energy axis; the shape factor is proportional to $1-(0.061\pm 0.015)q^2$. In this case the constants defined in section 4 are $d = 1.174$,

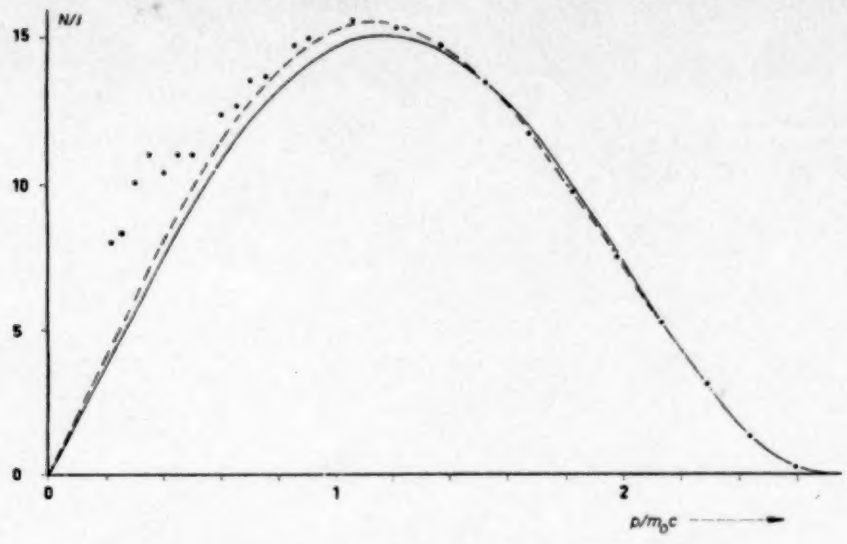

Fig. 5. Beta spectrum of ^{198}Au. The drawn line is the extrapolation from the straight line Fermi plot in fig. 3; the dotted line is the extrapolation using the analysis of fig. 4.

$e = 0.52$, $f = 0.29$, $\xi = 13.27$ (found from an analysis of the beta decay functions given by Rose *et al.*[15]). Introducing these values in eq. (4.7) (the transition under consideration is a $J = 1$ transition) we find the following relation between the matrix elements playing a role in this decay:

$$v_{na}-2.5a_n-1.5v_n = 0.$$

The matrix element ratios themselves cannot be determined from this type of experiment except under the following conditions:

A) Cancellation must occur, so that the above approximation is not valid. Usually $S_{1,2}$ is then comparable to $S_{1,1}$ so that the matrix element a_{nu} becomes important, unless

B) the transition under consideration is a 1—0 transition.

Both conditions are fulfilled in the decay of RaE; this explains that in RaE Wu [2]) could give separate values $t_{1a}/t_1 = 1.51$ and $s_1/t_1 = 0.22$. It is easily checked that with these values S_{12} becomes comparable to S_{11}.

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CONVERSION COEFFICIENTS OF GAMMA TRANSITIONS IN ^{203}Tl

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Abstract: Conversion coefficients in ^{203}Tl are determined as $\alpha_K = 0.163 \pm 0.003$, $\alpha_L = 0.0487 \pm 0.0012$ for the 279.12 keV gamma transition, and $\alpha_K = 0.118 \pm 0.011$ for the 404 keV transition. These results are discussed in view of possible deviations from theoretical values for M1 and E2 conversion coefficients. The decay energy of ^{203}Pb is estimated to be 950^{+300}_{-70} keV.

1. Introduction

Measurements of K and L conversion coefficients of M1 transitions in nuclei with mass number around 200 indicate values significantly lower than theoretical results computed by Rose [1]), in the point nucleus approximation, as shown in a paper by Wapstra and Nijgh [2]). This discrepancy is explained as an influence of the nuclear structure as predicted by Sliv and Listengarten [3]). The present paper gives details of the measurements of the conversion coefficients of the 279 and 404 keV transitions in ^{203}Tl discussed in ref. [2]). Fig. 1 shows the decay schemes of ^{203}Hg and ^{203}Pb leading to the levels under consideration [4, 5, 6]).

Measurements of Stockendal *et al.*[7]) on the decay of the isomer in ^{203}Pb indicate strongly that the ground state in this nucleide is $f_{\frac{5}{2}}$, confirming earlier assignments by Prescott [4]) and Varma [5]). The best energy values for the gamma rays are 279.12 ± 0.05 keV [8]) and 403.8 ± 0.3 keV [9]). Table 1 summarizes earlier measurements concerning the conversion coefficients of the 279 keV transition (those for the 404 keV transition are discussed in section 7), together with the results obtained in the present work.

Part of the measurements discussed in this paper have already been described in a thesis by Nijgh [10]).

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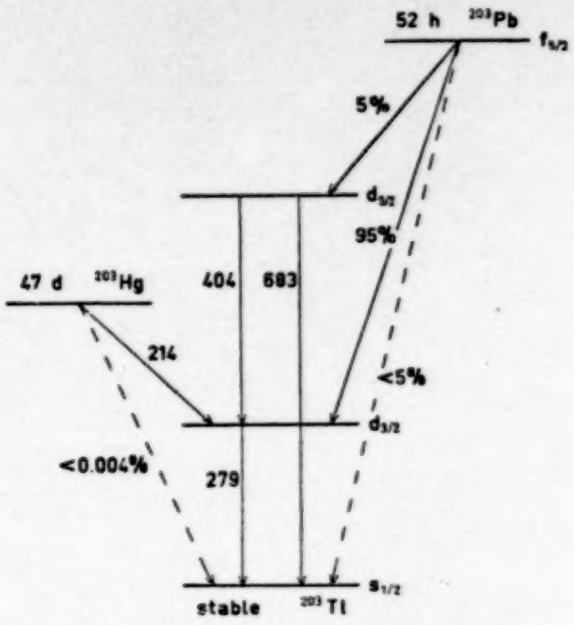

Fig. 1 Decay schemes of ^{203}Hg and ^{203}Pb.

2. Sources

^{203}Pb sources were made in the same manner as described earlier [6]).

The first ^{203}Hg source (I) was prepared by mass separation in the Gothenburg isotope separator [11]) onto a 200 μg/cm^2 thick Al foil. The charge material to the separator was mercury oxide activated by neutron irradiation at Harwell. This source was rather weak. In order to obtain stronger sources, a neutron irradiation of metallic Hg was made in the high flux reactor in Arco, Idaho. This sample was purified by dissolving in nitric acid and precipitating mercury oxide with sodium hydroxide. Part of this material was then dissolved in nitric acid and used to prepare a beta spectrometer source (II) by the insuline spread technique on an aluminized formvar foil made acid resistant with a thin polystrene cover. The average thickness of this source was estimated to be 30 μg/cm^2, its backing 20 μg/cm^2. Source III was made from the same material using the mass separation technique of source I; in this case the mass separation was carried out in the Copenhagen isotope separator. Source IV was mass separated in Gothenburg and collected on a 200 μg/cm^2 Ag backing. All samples were covered by a 6 μg/cm^2 formvar foil.

3. The Beta Spectrometer

We have used the iron free short lens beta-ray spectrometer described by Verster [12]) and by Nijgh [10]). The transmission of instruments not containing

TABLE 1

The beta energy of ^{203}Hg and conversion coefficients of the 279 keV transition in ^{203}Tl

Author	E_β(keV)	$\alpha_K \times 10^3$	K/L	L/(M+ ...)	K/(L+M+ ...)
a)	205±10	180	3	≧12	
b)	208	195 ††	3	≪12	
c)	210	195			3.7
d)		230±10			
e)	210±2	154±15			2.78±0.20
f)		163±6 †††	3.2 †††	3.2 †††	2.5 †
g)	219±10	141±15	3.30±0.10	4 ±1	2.57±0.07
			3.30±0.10 †	3.5 ±0.3 †	2.55±0.05 †
h)	214±2	205±10	3.15±0.15	4.5 ±1.5	
i)		147±2	3.29		
j)		159±4	3.24±0.05		
k)			3.42		
l)		150±10			
m)	195±2	130±10			2.75
n)	225	160			2.80
o)		140 †			
present work	214±3	163±3	3.39±0.06	3.28±0.10	2.60±0.06
			3.34±0.10 †	3.40±0.15 †	2.60±0.07 †

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†) The results marked † have been obtained from studies of ^{203}Pb; all other results are from observations on ^{203}Hg
†† Computed from given α_{tot} and K/(L+M+ ...) = 2.6
††† Private communication from F. R. Metzger to J. Varma

ferromagnetic materials is independent of the energy setting if the earth's magnetic field is compensated or if it can be neglected. Iron free beta spectrometers therefore need not be calibrated for intensity comparisons. (In instruments containing iron, remanents magnetism and saturation effects may spoil this energy independence.) Only the earth field component normal to the spectrometer axis has been compensated in our instrument. The axial component is, however, at most only a few percent of the average magnetic field in our spectrometer, and therefore does not significantly

influence intensity measurements; energy determinations are corrected for its presence.

The electrons are detected by a Geiger-Müller counter with a formvar end window 40 μg/cm^2 thick or less, supported by 0.02 mm thick nylon wires. These wires cover about 4 % of the window surface. A range of 0.02 mm in nylon corresponds to 30 keV electrons; hence, the window transmission will not vary more than about 1 % for electron energies above 60 keV. The low energy cut-off of the counter, if filled with a mixture of 5 cm Hg of argon and 1 cm of alcohol vapour, occurs at about 5 keV. The counter is used in combination with a den Hartog [13]) quenching circuit with constant dead time. Usually the dead time was 3 msec; when high counting rates occurred a 0.7 msec dead time was sometimes used. Results obtained with different dead times agreed well. We have assumed the counter efficiency to be independent of the electron energy.

4. General Remarks about the Intensity Measurements

The best measure for the intensity of a spectral line or a beta continuum as obtained with an iron free magnetic beta-ray spectrometer is its area in a plot of N/i versus i, (or N versus log i), N being the counting rate and i the spectrometer current (corrected for the remaining earth field component, if any).

Completely satisfactory measurements of the beta continuum could not be obtained with our equipment, as can be seen from one of the Fermi plots (fig. 2). The 279 keV conversion line at 193 keV distorts the spectrum at energies above 180 keV. The Auger lines contribute to the spectrum at energies around 55 keV and 70 keV. We always find apparently too many electrons in the energy range below 50 keV. This is partly due to back-scattering of electrons in the source backing, as is evident from the fact that the excess with source IV (silver backing) was slightly larger than that with source III (aluminium backing). A still smaller excess was found with source II (thin formvar backing) but this may also partly be due to absorption in this somewhat thicker source. Nevertheless with all sources the excess was nearly 15 % of the number of electrons. We are inclined to ascribe this effect to scattering in the spectrometer, the more so since a similar excess was found in measurements of the ^{198}Au beta spectrum with almost ideal sources [14]).

The intensity of the beta spectrum was therefore always obtained by extrapolation from measurements at $\approx$ 60 keV and between 85 and 180 keV, assuming a linear Fermi plot, and employing the Fermi functions given by Fano [15]), corrected for screening using the data computed by Reitz [16]) and Flügge and Jekeli [17]). This procedure is not quite correct for this first forbidden transition: the Fermi plot should, according to the analysis given by

Wapstra [18]), have a shape $(1+gq)q$ (q being the neutrino energy W_0-W). Our measurements, analysed in the way proposed by Wapstra [18]), only indicate that g should lie between 0 and 0.2: the latter value for g would yield a 3 % larger intensity for the beta continuum (and therefore a 3 % lower value for the conversion coefficient) than $g = 0$. A value of g as large

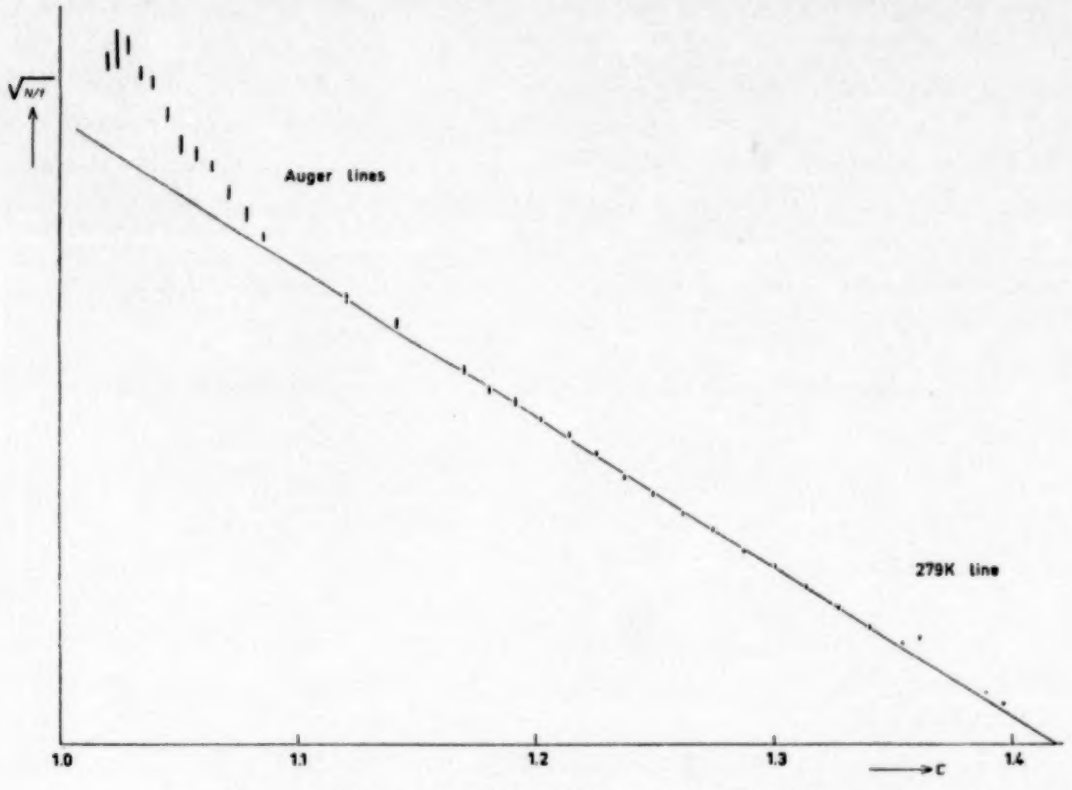

Fig. 2. Fermi plot of the beta spectrum of ^{203}Hg.

as 0.2 would, however, only be obtained if the matrix elements for this transition would cancel to a considerable extent; this disagrees with the fact the ft value for this transition is normal for a first forbidden spectrum ($\log_{10} ft = 6.4$). We therefore assumed $g = 0$; the error introduced by this approximation is estimated to be less than 1 %.

The comparison between the intensities of the electron lines and the recomputed beta spectrum was made by plotting them, preferably overlapping, on one piece of graph paper and measuring their surfaces with a planimeter. The error introduced in this way is estimated to be less than $\frac{1}{2}$ %.

5. Measurements of the 279 keV Line

New measurements on ^{203}Pb were made with a line halfwidth of 0.8 %. The L and M lines were now almost completely separated; a better value could therefore be given for the L/M ratio of the 279 keV line (see table 1).

The ^{203}Hg source I (see section 2) was measured with 2.4 % line halfwidth. Four independent runs yielded an average value of 0.137 for the ratio of the K conversion line to the recomputed beta continuum; this result corresponds to a K conversion coefficient of 0.170 ± 0.007. The error is almost entirely due to statistics, the maximum intensity of the beta continuum being only a few times background.

Sources II and III were measured with line widths of 1.7 % and 0.8 % respectively, yielding an average conversion coefficient of 0.163 ± 0.003 as mentioned in table 1. The 279 K line in source III was measured 8 times during a 2 weeks' interval with both Geiger counter dead times; the results agreed very well. The beta spectrum and the L and M lines were measured 4 times in the same period. The statistical errors in the intensities are less than 1 % in all cases.

Source IV was only used for investigating the low energy excess in the beta continuum as discussed in section 4.

6. Discussion of the Conversion Data for the 279 keV Transition

Theoretically, the internal conversion coefficient of a mixed M1—E2 transition in an electron shell $X(= K, L_I, L_{II} \ldots)$ may be expressed as a linear combination $a\beta_1^{(X)} + (1-a)\alpha_2^{(X)}$ of the corresponding coefficients

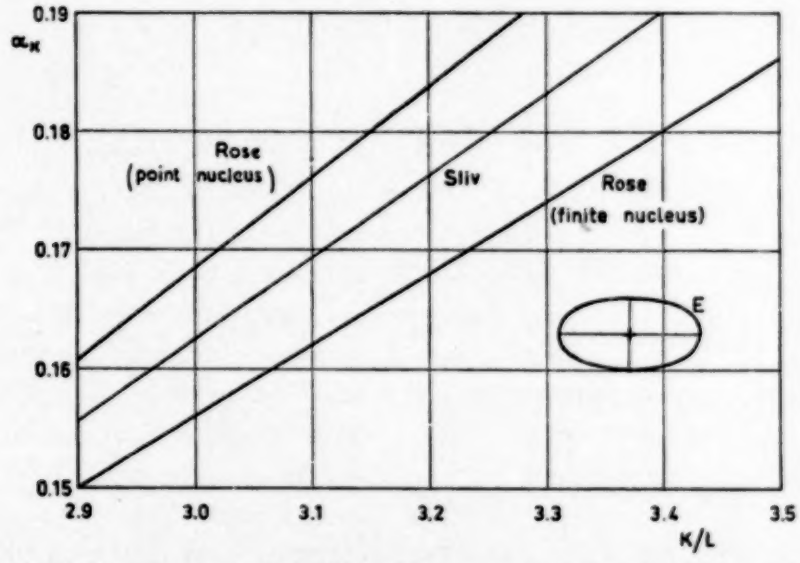

Fig. 3. Relation between theoretical K conversion coefficients and K/L ratio for a mixed M1—E2 transition of 279 keV in Tl, according to Rose and to Sliv and Band. Our experimental values have been indicated by E.

$\beta_1^{(X)}$ and $\alpha_2^{(X)}$ for pure magnetic-dipole and pure electric-quadrople radiation. The constant a represents the percentage of the gamma rays that have M1 character and therefore is the same for all electron shells X. Elimination

of a then gives a relation between the K shell conversion coefficient α_K and the K/L ratio, which is depicted in fig. 3.

As discussed earlier [2]) the experimental values, also shown in fig. 3 (point E), cannot be reconciled with the relation between α_K and K/L as computed from Rose's point nucleus values. This disagreement is at least partly due to the effect of the nuclear structure on the conversion coefficients. The influence of this effect has first been calculated by Sliv and Band [3,19]), assuming a nuclear model with a homogeneous charge distribution with a radius of $1.2\times10^{-13}\times A^{\frac{1}{3}}$ cm and with surface currents. These authors estimate that their values will not be altered by more than 2 % if their nuclear model were replaced by a more realistic one. Nevertheless, Sliv and Band's values do not agree much better with the experimental results than Rose's point nucleus values (see fig. 3).

Green and Rose [20]) separated the influence of the nuclear structure in a static and a dynamic part. Conversion coefficients only corrected for the static part have been computed by Rose [21]). These values agree much better with our experimental results (see fig. 3). According to Kisslinger [22]), the dynamic finite size effect would, in the case under consideration, decrease the M1 K conversion by 10 to 20 %, which would even improve the agreement. As discussed elsewhere [23]), the effects mentioned above do not, however, offer a very acceptable explanation for the L subshell ratios.

Deviations from Rose's conversion coefficients and from those of Sliv and Band similar to those mentioned above have been reported by Nielsen *et al.*[24]) and by Stelson and McGowan [25]).

7. The 404 keV Transition

Earlier measurements [6]) yielded a value 0.037 ± 0.002 for the intensity ratio of the 404 K and 279 K conversion lines in the decay of ^{203}Pb. This result disagreed with the value 0.020 computed from the data given by Varma [5]). New measurements with our beta spectrometer set to a half-width of 1.1 % yielded a ratio 0.037 ± 0.003, confirming our earlier value.

The ratio of the gamma intensities was remeasured in a scintillation spectrometer calibrated for intensity measurements by comparing 279 keV and 412 keV gamma rays in ^{203}Hg and ^{198}Au, respectively, with the KX-rays due to internal conversion in these same samples; we find $279\gamma/404\gamma = 19.5\pm1.5$. This result agrees with the value 21.3 ± 1.5 given by Prescott [4]) and with our own earlier value [6]) which, after changes in the calibration necessitated by the new values of the ^{203}Hg and ^{198}Au conversion coefficients, comes out to be 18.5 ± 2. Together with the conversion line ratio discussed above the K shell conversion coefficient of the 404 keV transition becomes 0.118 ± 0.011. This value agrees well with the earlier value 0.12 used in our previous paper [2]).

8. The Decay Energy of ^{203}Pb

In an earlier paper [6]) we computed this quantity from the experimental number of K capture transitions per decay of ^{203}Pb using a formula given by Marshak [26]) which, however, is only correct for unique transitions. Formulae applicable for non-unique transitions are given by Brysk and Rose [27]). The ratio of L and K capture in a non-unique first forbidden transition is

$$\frac{\varepsilon_L}{\varepsilon_K} = \frac{g_{L_I}^2}{g_K^2}\left(1+\frac{f_{L_{II}}^2}{g_K^2}\right)\left(\frac{\Delta - B_L}{\Delta - B_K}\right)^2.$$

Δ is the transition energy, B_K and B_L are the electron binding energies in the K and L electron shells; the values of the wave function component ratios in Tl are $g_{L_I}^2/g_K^2 = 0.152$ and $f_{L_{II}}^2/g_{L_I}^2 = 0.083$ [27]). Although the values of the nuclear matrix elements in the transitions have some influence, rather extreme combinations of matrix elements are necessary in order to change $\varepsilon_L/\varepsilon_K$ by more than 1 % from the value computed using the above constants. Capture in the L_{III} shell can be neglected. According to Adams [28]) the ratio of capture in higher shells is $\varepsilon_M/\varepsilon_L = 0.21$; $\varepsilon_N/\varepsilon_L = 0.05$ etc. for $Z = 80$ and $\Delta = \infty$; using estimates for capture in still higher shells and an approximate value for the decay energy we find $(\varepsilon_L+\varepsilon_M+\ldots)/\varepsilon_K = (1.27\pm0.01)\varepsilon_L/\varepsilon_K$ Taking all these data together,

$$\frac{\varepsilon_L+\varepsilon_M+\cdots}{\varepsilon_K} = 0.209\left(\frac{\Delta - 15\text{ keV}}{\Delta - 85\text{ keV}}\right). \qquad (1)$$

Information about the decay energy can be derived from a comparison of the measured percentage K capture with the theoretical formula. Table 2

TABLE 2

Capture ratios in the decay of ^{203}Pb

Author	Result	Computed decay energy to ground state (keV)
Prescott [4])	$\varepsilon_K^{(1)} : \varepsilon_{tot}^{(1)} = 0.87\pm0.05$	> 3400
	$\varepsilon_K^{(2)} : \varepsilon_{tot}^{(2)} = 0.74\pm0.05$	1000^{+310}_{-90}
	$[\varepsilon_K^{(2)} : \varepsilon_{tot}^{(2)}]/[\varepsilon_K^{(1)} : \varepsilon_{tot}^{(1)}] = 0.85\pm0.07$	900^{+80}_{-20}
Present paper	$[\varepsilon_K^{(0)} + \varepsilon_K^{(1)} + \varepsilon_K^{(2)}]/[\varepsilon_{tot}^{(1)} + \varepsilon_{tot}^{(2)}] = 0.79\pm0.02$	$970^{+600\,\dagger}_{-140}$

† assuming $\varepsilon^{(0)} = 0$.

On the basis of these data a value 950^{+300}_{-70} keV is proposed for the decay energy, as discussed in the text.

summarizes the available data; we have indicated by $\varepsilon_K^{(1)}$ and $\varepsilon_{tot}^{(i)}$ the intensities of K capture decay and total capture decay to the state i;

$i = 0, 1, 2$ referring to the ground state and the 279 keV and 683 keV levels respectively. The result marked "present paper" is derived from an earlier measurement [6]) of the ratio $(X_K : 279\gamma)_{^{203}Pb}/(X_K : 279\gamma)_{^{203}Hg}$, together with the more accurate data on the transitions in ^{203}Pb reported in this paper.

Prescott reports total decay energies 730 keV and 910^{+70}_{-40} keV, following from his capture ratios to the first and second excited states respectively. In deriving these values he has, however, neglected the occurrence of capture in the M, N, . . . shells. If this effect is taken into account, as has been done in eq. (1), one obtains the values quoted in table 2, which do not agree very well. A decay energy may also be derived from Prescott's double ratio

$$(\varepsilon_K^{(2)} : \varepsilon_{tot}^{(2)})/(\varepsilon_K^{(1)} : \varepsilon_{tot}^{(1)}).$$

The value derived in this way should be more accurate than those found from the single ratios since it is less influenced by systematic errors affecting both results in the same way. Considering Prescott's data together we estimate the ^{203}Pb decay energy 950^{+300}_{-70} keV, corresponding with capture ratios

$$\varepsilon_K^{(1)}/\varepsilon_{tot}^{(1)} = 0.792^{+0.012.}_{-0.005}, \qquad \varepsilon_K^{(2)}/\varepsilon_{tot}^{(2)} = 0.72 \pm 0.07;$$
$$(\varepsilon_K^{(2)} : \varepsilon_{tot}^{(2)})/(\varepsilon_K^{(1)} : \varepsilon_{tot}^{(1)} = 0.90 \pm 0.08.$$

Our own result mentioned in table 2 can only be used to determine a decay energy if some assumption is made about the intensity of the ground state transition. The decay energy given in the table has been computed assuming this intensity to be negligible; if a seizable ground state transition is present the decay energy will be less. Conversely, adopting our above estimate for the decay energy, our result yields a value (0 ± 3) % for the total intensity of electron capture transitions directly to the ^{203}Tl ground state. This result is to be compared with the value (-6 ± 5) % reported by

TABLE 3

K capture transitions in ^{203}Pb

Decay energy	Intensity	Reduced halflife	Level assignments
950 keV	< 5 %	$\log_{10} f_1 t > 7.3$	$f_{\frac{7}{2}} \rightarrow s_{\frac{1}{2}}$
670 keV	0.792 × 95 %	$\log_{10} f_0 t = 6.4$	$f_{\frac{7}{2}} \rightarrow d_{\frac{3}{2}}$
270 keV	0.72 × 5 %	$\log_{10} f_0 t = 6.9$	$f_{\frac{7}{2}} \rightarrow d_{\frac{5}{2}}$

Varma, and a value (-4 ± 5) % that can be derived from fig. 5 of Prescott's paper. Taking all results together, we think that it can safely be assumed that less than 5 % of the decays of ^{203}Pb proceed directly to the ^{203}Tl ground state.

Table 3 shows the reduced half-lives for the K capture transitions under consideration, computed from the above data. They check well with the proposed level assignments.

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CONVERSION COEFFICIENTS OF GAMMA TRANSITIONS IN ^{198}Hg

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Abstract: The K and L conversion coefficients of the 412 keV E2 transition following the decay of ^{198}Au are measured to be 0.0281 ± 0.0005 and 0.0097 ± 0.0004. Both values are about 10 % lower than the theoretical values. The main beta spectrum of ^{198}Au agrees with the theoretical shape for first forbidden non-unique spectra; its endpoint is 966 ± 3 keV.

1. Introduction

Recent experiments [1,2,3]) show that conversion coefficients of M1 gamma rays may deviate somewhat from the theoretical values computed by Rose [4]) and by Sliv and Band [5]). This deviation may be due to the dynamic finite

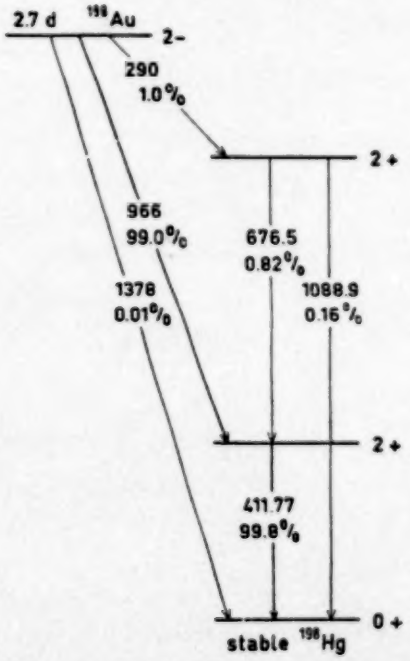

Fig. 1. Decay scheme of ^{198}Au.

size contribution [6]) which has not been included in Rose's computation and only approximately in those of Sliv and Band. This fact may explain the

differences between their values for M1 conversion coefficients. Their E2 conversion coefficients agree very well; indeed the dynamic finite size contribution is expected to be smaller in this case. It is therefore of interest to compare also some E2 conversion coefficients with the theoretical values. With this purpose in mind, we have tried to measure accurately the conversion coefficients in ^{198}Au. Fig. 1 shows the decay scheme of this nucleide. An accurate measurement in a beta-ray spectrometer of the intensity of the conversion lines relative to the intensity of the continuous beta spectrum

TABLE 1

Conversion coefficients ($\times 10^4$) of the 412 keV gamma transition in Hg^{198}

Ref.	α_K	α_L	$\alpha_{L+M+\ldots}$
[7])	250		
[8])	350		135
[9])	300	100	130
[10])	280±70	125±30	155±40
[11])	210	100	127
[12])	318	103	134
[13])	300		130
[14])	275±30		125
[15])	277	100	135
[16])	250±50	120±20	
this work	281± 5	97± 4	128± 3
theoretical	302	110	

will give the conversion coefficients. Table 1 presents the results of earlier measurements of the conversion coefficients of this transition, the results of the present investigation and the theoretical values.

2. Active Sources; the Beta-Ray Spectrometer

The active material was obtained from Messrs. Philips-Roxane N.V.; it had been prepared by neutron irradiation of gold and was chemically purified. At the time of our measurements the specific activity amounted to nearly 10 curies per gram. The beta-spectrometer sources were prepared by vacuum evaporation of gold chloride onto a thin formvar backing made conducting by vacuum deposition of a thin layer of silver. The best average source thickness obtained was 8μg/cm^2; the thickness of the backing material was estimated to be less than 40 μg/cm^2. The high quality of the source can be seen from the appearance of the L Auger line group (6—14 keV) in the beta spectrum (see fig. 2.).

The properties of the iron free lens beta-ray spectrometer used in the investigation are described in the preceding paper [17]).

3. The Continuous Beta Spectrum

The sources contained about 1.5 % ^{199}Au formed by double-neutron capture, as was determined by comparing conversion line intensities. Our measurements were corrected for the presence of this impurity using the available information [18, 19, 20]) about its decay scheme. We also corrected our results for the presence of the 290 keV beta-ray component of 1 % intensity in the continuous spectrum of ^{198}Au; a correction for the very weak ground state transition is not necessary. The total correction was less than 5 % in the counting rate for each spectrometer setting. The resulting intensity data are plotted in fig. 2. The conversion lines and the K Auger lines are not

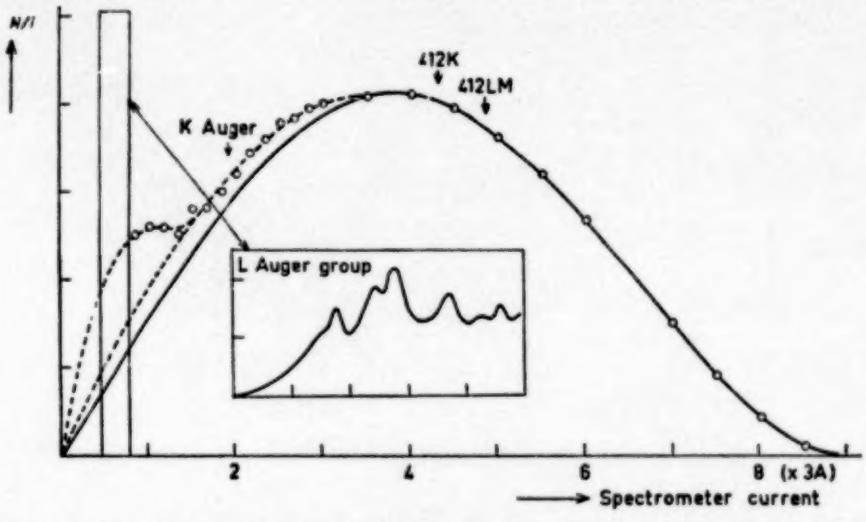

Fig. 2. Main beta spectrum of ^{198}Au. A correction has been applied for the beta spectrum of ^{199}Au and for the 290 keV beta spectrum of ^{198}Au. Insert: L Auger Lines (not corrected for counter window absorption).

shown. These results were analysed by making use of the values for the Fermi function f tabulated by Fano [21]) corrected for screening by the data given by Reitz [22]) and Flügge and Jekely [23]). The theoretical energy distribution in a first forbidden beta spectrum can be written [24]) as

$$N(p)\mathrm{d}p = (W-W_0)^2 f\{1+g(W_0-W)\}^2\,\mathrm{d}p.$$

The constant g in this formula is a quantity which shows the deviation from a straight-line Fermi plot ($g = 0$); it has to be determined experimentally.

Plotting $(N/f)^{\frac{1}{2}}(W_0-W)^{-1}$ as a function of W, one determines by trial and error the value W_0 for which this plot approaches a straight line (see fig. 3). We find $W_0 = 2.890$ (giving $E_\beta = 966\pm3$ keV) and $g = 0.071$. An experiment employing a thicker source and source backing [24]) yielded $W_0 = 2.892$, $g = 0.058$. Though the higher value $g = 0.071$ is obtained with the thinner source we do not consider it to be more reliable; therefore we shall use an average value of 0.065 ± 0.01.

No significant deviation from linearity in the Fermi plot has been reported before [12, 25–30]). This may have been due to the source thickness, which in our case has been at least twenty times smaller than those used earlier.

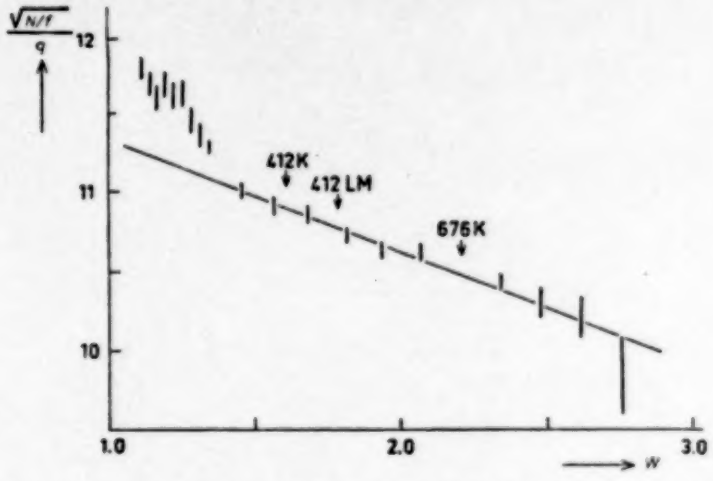

Fig. 3. "First forbidden" plot of the main beta spectrum of ^{198}Au.

Moreover, the deviations of our experimental points from a straight-line Fermi plot with a maximum energy of 958 keV are only of the order of 2 % for energies above 200 keV. Our result therefore agrees reasonably well with earlier measurements which gave maximum energies of 960±5 keV [7]), 956±5 keV [27]), 959±2.5 keV [28]) and 960±2 keV [29]). Fig. 2 shows the main ^{198}Au beta spectrum recomputed from this analysis. Breaks in the experimental curve occur at 200 keV and 40 keV. Therefore, two distorting factors appear to be present in our measurement, showing a spectrum of excess electrons with a maximum energy just below the top of the main beta spectrum, and another extending up till 40 keV, both of 5 % intensity. Measurements on the 210 keV continuous beta spectrum of ^{203}Hg in the same instrument [17]) gave an excess for electron energies lower than 50 keV; the magnitude of this excess varied with the source backing (200 μg/cm^2 Al and 200 μg/cm^2 Ag) but was of the order of 15 %. This excess may be due to the same influences as described above, which in the ^{203}Hg case cannot be separated; its somewhat larger magnitude (15 % versus 10 % may be ascribed to the thicker backing of the ^{203}Hg sources. In both cases the excess electrons are considered as arising from scattering effects.

4. The 412 keV Transition

Fig. 4 shows the conversion lines of the 412 keV transition, the width at half-maximum being 1.8 %. At least 10^4 counts per point have been taken. The K conversion coefficient is found to be 0.0280. An earlier measurement

with a thicker (20 μg/cm^2) source yielded 0.0282. The conversion coefficient presented in table 1 is the average of both measurements. In estimating the assigned error we assume that the continuous spectrum shape formula used

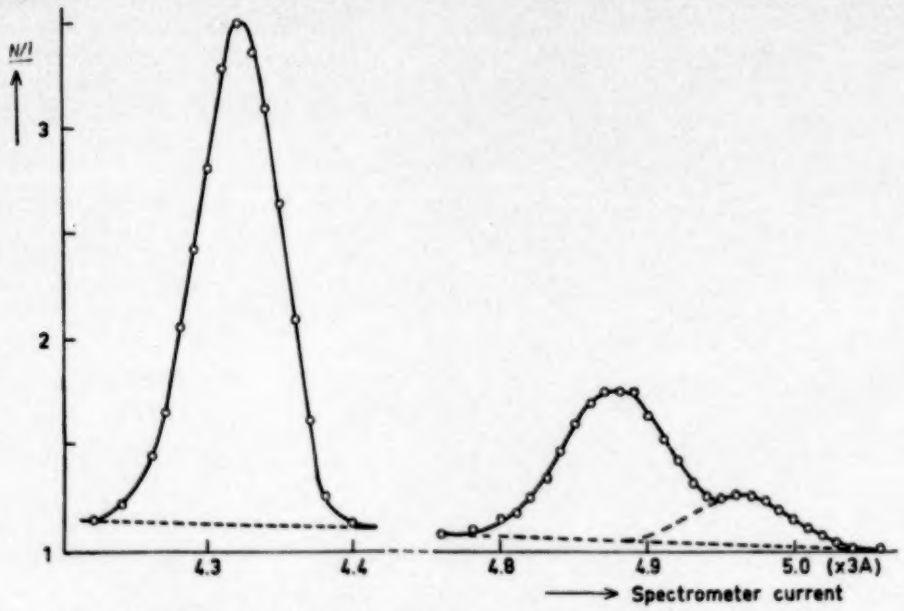

Fig. 4. Conversion lines from the 412 keV transitions.

above is exact; corrections for the line shape were found to be negligible. A Fermi analysis ($g = 0$) would have yielded the value 0.0288 for the K conversion coefficient; the result 0.0254 is found if the spurious electron spectra discussed above are included in the continuous spectrum. The L and M+N conversion lines were not completely separated in our measurements; therefore, the L conversion coefficient (table 1) is less precisely determined than the L+M coefficient.

Our results agree reasonably well with most other data (compare table 1). The K and L conversion coefficients are smaller by factors 0.93 ± 0.02 and 0.88 ± 0.04 than Rose and Sliv's theoretical E2 conversion coefficients (see table 1).

5. Other Transitions in ^{198}Hg

The conversion coefficients of the 677 keV and 1089 keV transitions in ^{198}Hg following the decay of ^{198}Au have been measured by Elliott *et al.*[28]) and Volpe and Hinman [31]) by a comparison of the gamma and conversion line intensities with those of the 412 keV transition; for the last transition they adopt a value $\alpha_K = 0.0317$. Table 2 shows their results corrected using our new value $\alpha_K = 0.0281$ for this conversion coefficient.

We have measured the intensity ratio of the 412 and 677 keV gamma rays in a scintillation spectrometer; our result 0.0083 ± 0.008 agrees well with the value 0.0082 ± 0.0006 given by Elliott *et al.* but is somewhat lower than

Volpe and Hinman's value 0.0097 ± 0.0011, indicating that their value for $\alpha_K(677)$ may be somewhat low.

In computing theoretical conversion coefficients, the 1089 keV transition is assumed to be pure E2 radiation; the 677 keV transition is taken to be a mixture of (40 ± 5) % M1 and (60 ± 5) % E2 [32,33]). The possibility of conversion of E0 character [6]) in the latter transition has not been taken into

TABLE 2

K conversion coefficients ($\times 10^4$) of transitions in Hg^{198}

Ref.	$\alpha_K(677)$	$\alpha_K(1089)$
Elliot [28])	198 ± 17	40 ± 3
Volpe [31])	169 ± 45	41 ± 5
Theoretical	203 ± 15	42

The experimental values have been corrected using the value $\alpha_K(412 = 0.0281$; the theoretical value for $\alpha_K(677)$ is computed for a mixture of (40 ± 5) % M1 + (60 ± 5) % E2.

account; a contribution due to this effect would increase its conversion coefficient. It is seen that, after making a correction using the new value for $\alpha_K(412)$, the conversion coefficients agree with the theoretical values within their errors, which are of the order of 10 %.

We thank Professor P. C. Gugelot for his interest in this work and Messrs. Philips-Roxane N.V. for the gold samples. We also wish to express our sincere gratitude to the staff of the Zeeman Laboratory of the University of Amsterdam for the hospitality extended to us during the many years that our beta ray spectrometer was accommodated there.

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CONVERSION COEFFICIENTS FOR ELECTRIC QUADRUPOLE TRANSITIONS

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Abstract: The measured K and L shell conversion coefficients of the 412 keV transition in ^{198}Hg and the 279 keV transition in ^{203}Tl are compared with the results of calculations by Rose and by Sliv and Band which are corrected for effects due to the finite nuclear size. The experimental results do not completely agree with the theoretical values, although neither E2 transition is delayed.

1. Introduction

Recently, conversion coefficients have been computed taking into account effects due to the finite nuclear size. Rose's tables [1]) do not take into account the dynamic effects of the nuclear structure as defined and treated by Green and Rose [2]); Sliv and Band's results [3]) include these effects only in a general way by adopting a simplified nuclear model. This may explain why Rose's and Sliv and Band's results for M1 transitions differ rather much, as shown in table 1 for the Th F line, and also why the latter results agree somewhat better in this case with the experimental values [4, 5]).

TABLE 1

Conversion data for the Th F-line ($E = 238.6$ keV, $Z = 83$)

	Rose	Sliv	Experimental
$10^4\alpha_K$	7600	7210	6700 ±500
$10^4\alpha_{L_I}$	1250	1170	1140 ±100
$10^4\alpha_{L_{II}}$	145	125	115 ± 13
$10^4\alpha_{L_{III}}$	9_3	8_8	
K/L_I	6.9	6.2	5.9± 0.3
L_I/L_{II}	10.1	9.3	9.9± 0.7
L_I/L_{III}	132	135	

In a few cases [4, 6]) M1 conversion coefficients deviate from the results of Rose and of Sliv and Band, which may again be explained by the dynamic finite size effects. If this explanation were correct, one would expect that in undelayed E2 transitions the conversion coefficients would agree much better with the theoretical values, since in these cases the dynamic finite size effects are expected to be small. (The last fact also explains that only minor

differences occur between Rose's and Sliv and Band's conversion coefficients for E2 transitions: see tables 2 and 3.) Below, we will present evidence that, nevertheless, conversion coefficients for some of these transitions are of the order of 10 % lower than the theoretical values. Deviation of a similar magnitude but opposite direction have been reported by Stelson and McGowan [7]).

2. The 412 keV Transition in ^{198}Hg

The 412 keV transition occurring in the decay of ^{198}Au represents a pure quadrupole transition since it leads to the 0^+ ground state in ^{198}Hg. The transition is not delayed [8]); dynamic effects of the nuclear structure should therefore not be significant [9, 10]).

TABLE 2

K and L shell conversion coefficients for the 412 keV E2 transition in ^{198}Hg (multiplied by 10^4)

Shell	K	L_I	L_{II}	L_{III}	L_{tot}
Rose	302	43.4	46.6	19.9	110
Sliv	301	42.7	46.8	19.7	109
experimental	281±5	37 ±4	46 ±5	15 ±2	97±4

Table 2 represents the K and L conversion coefficients determined by Wapstra *et al.*[11]), and the L subshell conversion coefficients derived from these results and the L subshell ratios determined by Reitmann *et al.*[12]). The same table reproduces the theoretical values [1, 3]), which agree well. The experimental results in the K and L shells are factors 0.93±0.02 and 0.89±0.04 lower than the theoretical values. There is no very significant difference in the behaviour of the three L subshells (correction factors 0.86±0.09, 0.98±0.11 and 0.76±0.10 for L_I, L_{II} and L_{III} respectively).

3. The 279 keV Transition in ^{203}Tl

The 279 keV transition following the decay of ^{203}Hg is a mixture of M1 and E2. The M1 part of the transition is somewhat delayed, the E2 part is not delayed [13, 14]). The following precise conversion data have been measured: $\alpha_K = 0.162\pm0.003$ [15, 16, 17]), $K/L = 3.37\pm0.05$ [15, 16, 17, 19, 20]), $(L_I+L_{II})/L_{III} = 4.82\pm0.15$ and $L_I/L_{III} = 1.54\pm0.03$ [18, 19, 20]). Table 3 gives the conversion coefficients derived from these data. Table 3 also presents theoretical conversion coefficients for pure M1 and E2 radiation as well as those for a mixture of these radiations with an intensity ratio as reported by McGowan and Stelson [21]). The last values deviate considerably from the experimental results. More-

over, a simple consideration makes clear that one cannot reach agreement by changing the multipole mixing ratio.

TABLE 3

K and L shell conversion coefficients for the 279 keV M1 +E2 transition in ^{203}Tl (multiplied by 10^4)

Shell		K	L_I	L_{II}	L_{III}	L_{tot}
Rose	M1	4110	586	55	5	647
	E2	764	108	243	127	478
	M1+E2†	1795	255	185	89	530
Sliv	M1	3960	626	64	5	695
	E2	765	108	252	128	488
	M1+E2†	1749	267	194	90	551
Experimental		1620±30	242±5	156±6	82±4	481

† For $\varepsilon^2 = 2.25$.

In order to analyse the situation, we apply correction factors $x^{(X)}$ and $y^{(X)}$ to the theoretical conversion coefficients in each shell X. The experimental results then give, for each shell, a relation

$$\alpha_{exp}^{(X)} = a x^{(X)} \beta_1^{(X)} + (1-a) y^{(X)} \alpha_2^{(X)} \qquad (1)$$

between x and y containing a parameter a which is the same for all shells ($a = (1+\delta^2)^{-1}$ if δ is the amplitude mixing ratio). Without further assumptions, this set of equations is difficult to analyse. A reasonable assumption seems to be that the L_{III} conversion coefficients agree with the theoretical results; the L_{III} electron wave function having a minimum at the nucleus, the L_{III} conversion coefficients will not much be affected by finite size effects. Then, eq. (1) taken for $X = L_{III}$ determines $a : a = 0.37 \pm 0.04$. Fig. 1 shows the relations between the x's and y's for this value of a and, for comparison, also for the value $a = 0.31$ corresponding to McGowan and Stelson's result $\delta^2 = 2.25 \pm 0.25$ [21]). It is now indeed clear that agreement with theoretical values can not be reached for any value of a, a variation of a that brings the K and L_1 lines closer to the point $x = 1$, $y = 1$ causes the L_I and L_{II} lines to move away from this point.

The results presented in fig. 1 show that, for any reasonable value of a, $y^{(L_{III})}$ and $y^{(L_{II})}$ are different. One has therefore to assume a deviation in the L_{II} or the L_{III} conversion coefficient for the E2 part of the transition. (The difference is more significant than is suggested by the errors indicated in fig. 1, since the conversion ratio L_{II}/L_{III} is better known than the conversion coefficients themselves.) Assuming $y^{(L_{III})} = 1$, as is likely if the deviation would be caused by finite size effects, $y^{(L_{II})}$ should be about 0.9, and therefore rather similar to the reduction factors as found in ^{198}Hg.

Without further assumptions, not much can be said about the other E2

reduction factors or the M1 ones. We have tried the set of assumptions

$$x^{(K)} = x^{(L_I)} = x^{(L_{II})} = x, \qquad x^{(L_{III})} = 1;$$
$$y^{(K)} = y^{(L_I)} = y^{(L_{II})} = y, \qquad y^{(L_{III})} = 1. \tag{2}$$

A least squares analysis starting from this assumption and Rose's conversion coefficients computed in the point nucleus approximation gave a perfectly

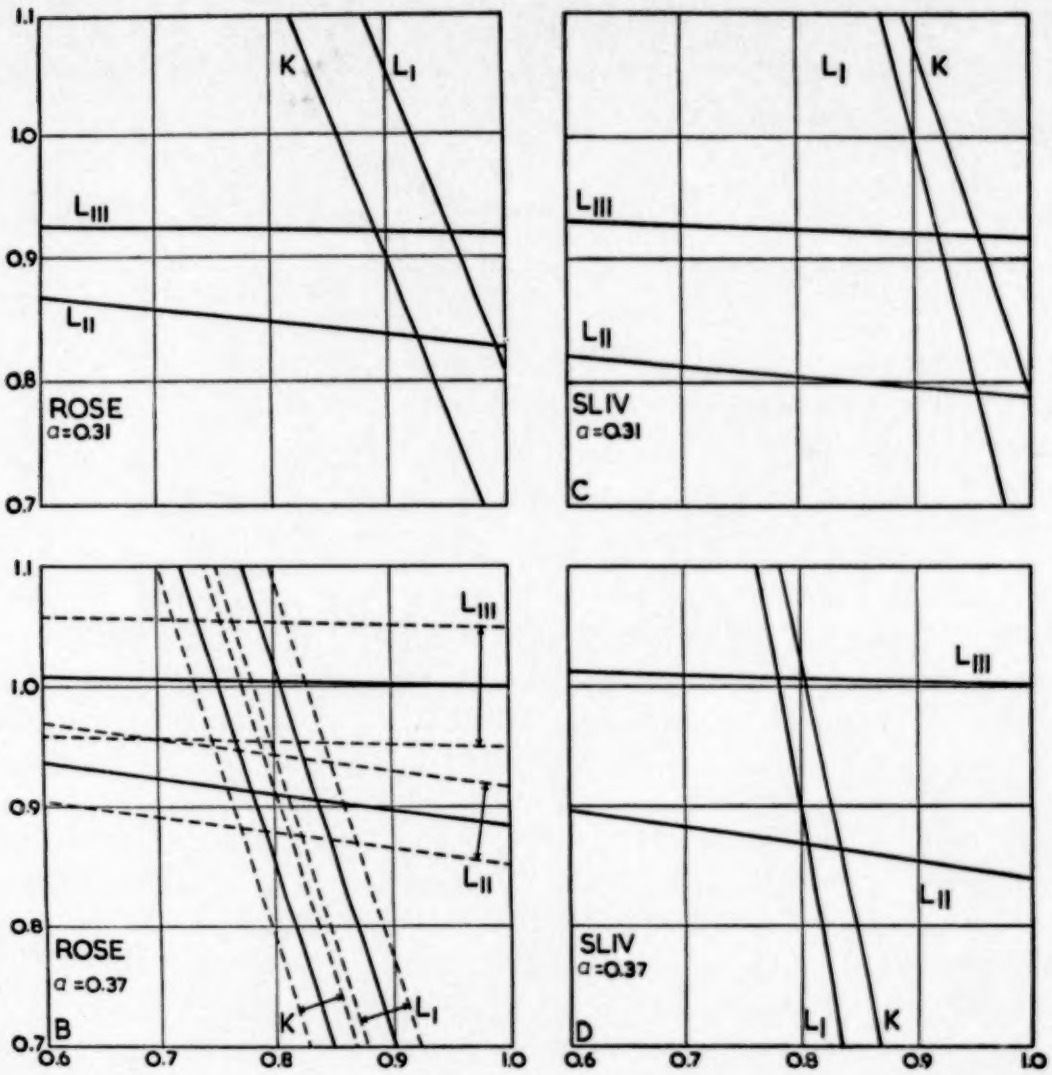

Fig. 1. Relations between the correction factors x and y to be applied to the theoretical M1 and E2 conversion coefficients for the K, L_I, L_{II} and L_{III} shells, following from the experimental data on the decay of the 279 keV level in ^{203}Tl. The diagrams have been drawn, for two adopted values of the percentage a of M1 gamma rays. Error bands are shown in one diagram; those in the other diagrams should be about equally wide.

consistent result [22]); using Rose's conversion coefficients corrected for the static finite size effect the consistency is not very good, the ratio of external to internal errors being 1.8. (The results using Sliv's values are quite analo-

gous.) This is perhaps not surprising since the assumptions (2) do not even hold for the ratios of Rose's and Sliv and Band's conversion coefficients. In addition, the value $a = 0.41 \pm 0.03$ derived from our analysis differs considerably from the result $a = 0.31 \pm 0.03$ derived from McGowan and Stelson's experiment [21]). The values $x = 0.71 \pm 0.06$ and $y = 0.91 \pm 0.03$ following from the least squares analysis should therefore be used with some care, though this value of x agrees reasonably with the estimate by Kisslinger [23]) that the dynamic finite size effect would decrease the M1 K conversion coefficient by an amount between 10 % and 20 %.

4. Conclusion

The above results indicate that in two E2 transitions conversion coefficients are about 10 % lower than Rose's and Sliv and Band's theoretical values. Since both transitions are undelayed, it is questionable whether these deviations can be explained by the dynamic finite size effect. The same conclusion applies to the results of McGowan and Stelson [7]). Unfortunately, the deviations are such that, anyhow in the case of Pb^{203}, they hamper an accurate determination of the multipole mixing ratio in M1+E2 transitions.

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ON THE DIRECT NEUTRON EXCHANGE INTERACTION OF COMPLEX NUCLEI INVOLVING A POSSIBLE LARGE CHANGE OF THE NUCLEAR SPINS

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Abstract: It is shown that it may be possible to observe a process of direct interaction of complex nuclei in which these nuclei, being on the outer boundary of the Coulomb potential barrier, exchange neutrons. If the neutrons in the outer shells have large momenta, their exchange may result in a large change of the spins of the interacting nuclei and in the excitation of otherwise inaccessible levels, corresponding to all possible values of the nuclear spin and its projection for different arrangements of a given number of neutrons in the outer shell.

The exponent characterizing the probability of the process under consideration is estimated according to a method proposed by L. D. Landau [1]) and applied to some other problems by E. M. Lifshitz [2, 3]).

1. Introduction

The acceleration of heavy ions achieved to date opens the possibility of observing a very extraordinary nuclear process: the direct exchange (i.e., an exchange not involving the formation of a compound nucleus) of neutrons between complex nuclei. The following are two possible examples of such a process:

1) $${}_{8}O^{16}_{8}{}_{0^+} + {}_{10}Ne^{20}_{10}{}_{0^+} \rightarrow {}_{8}O^{16*}_{8}{}_{0,1,2,3,4,5^+} + {}_{10}Ne^{20*}_{10}{}_{0,1,2,3,4,5^+}$$ (exchange of neutrons in the $d_{\frac{5}{2}}$ shell)

2) $${}_{36}Kr^{84}_{48}{}_{0^+} + {}_{36}Kr^{84}_{48}{}_{0^+} \rightarrow {}_{36}Kr^{84*}_{48}{}_{0,1\ldots8,9^+} + {}_{36}Kr^{84*}_{48}{}_{0,1\ldots8,9^+}$$ (exchange of neutrons in the $g_{\frac{9}{2}}$ shell)

It is evident that if the neutrons in the outer shells have large momenta their exchange can result in a very large change of the spins of the interacting nuclei, i.e., it can lead to the excitation of levels which differ greatly from the ground state in the magnitude of their angular momentum, but which are comparatively weakly excited ($\lesssim 1$ MeV). Therefore, it may be that this process, in addition to its specific nature, may also be of interest for the study of new, hitherto unknown nuclear levels, corresponding to all possible values of the nuclear spin and its projection for different arrangements of a given number of neutrons in the outer shell.

2. Estimate of the Exponent Characterizing the Probability of the Process

In order to estimate the probability of the process under consideration we shall resort to the general method proposed by L. D. Landau [1]) to describe transitions in quasi-classical systems and applied by E. M. Lifshitz to the interaction of deuterons with nuclei [2]) and to the transfer of neutrons from one complex nucleus to another [3]).

As is shown in ref.[1]), the probability w of transition of a quasi-classical system upon collision (without radiation) from an initial to a final state is proportional to the exponential factor

$$\phi = \exp\left\{-\frac{2}{\hbar}\,\mathrm{Im}\,[S_{\mathrm{I}}(\mathrm{I},\,\xi)+S_{\mathrm{II}}(\xi,\,\mathrm{II})]\right\} \qquad (1)$$

where $S_{\mathrm{I}}(S_{\mathrm{II}})$ is the action in the initial (final) state, I and II denote an initial and a final point, and ξ the point of transition, corresponding to the minimum of Im $(S_{\mathrm{I}}+S_{\mathrm{II}})$, i.e. to the largest value of w.

The application of (1) to the analysis of the process under consideration is exactly similar to that carried out in Lifshitz's paper [3]), with the only difference that we consider not one act of transfer of a neutron from nucleus 1 to nucleus 2, but two such acts (the transfer of neutron n_1 from 1 to 2 and of neutron n_2 from 2 to 1), both together representing the neutron exchange interaction.

We introduce the following notations:

t_i are the instants of neutron emission by nuclei $1(t_0)$ and $2(t_1)$ and the instants of neutron capture by nuclei $2(t_2)$ and $1(t_3)$;

ξ_i the distances between the nuclei 1 and 2 at the instants t_i;

x_{ci} the coordinates of the centre of mass of the system of nuclei 1 and 2 at the instants t_i;

${}_ix_{\mathrm{n1}}$, ${}_ix_{\mathrm{n2}}$ the coordinates of the neutrons (n_1 and n_2) at the instants t_i;

M_1, M_2, m the masses of the nuclei 1 and 2 and of the neutron;

Z_1, Z_2 the charges of the nuclei;

$\mu = M_1M_2/(M_1+M_2)$ and μ_{ik} the reduced masses of the nuclei in the initial and final states, and in the time intervals between t_i and t_k, respectively;

E_0, E_3, E_{ik} the energy of relative motion of the nuclei 1 and 2 at $t \leqq t_0$ for $t \geqq t_3$ and in the time intervals between t_i and t_k, respectively;

V_{ik} the velocity of the centre of mass of the two nuclei in the time intervals between t_i and t_k;

I_1, I_2 the binding energies of the neutrons emitted by the nuclei 1 and 2;

$I'_1I'_2$ the binding energies of the neutrons emitted by the nuclei 1 and 2 after preliminary capture of a neutron (as a rule, $I'_1 < I_1$ and $I'_2 < I_2$).

We considerably simplify all the calculations by assuming $m \ll M_1$ and $m \ll M_2$. It then follows from the laws of conservation of energy and momentum that if prior to neutron emission (i.e., at $t \leqq t_0$) the velocities of the nuclei 2 and 1 in the centre of mass system were

$$v_2 = \frac{M_1}{M_1+M_2} v_0, \qquad v_1 = -\frac{M_2}{M_1+M_2} v_0,$$

where $v_0 = v_2 - v_1$ is the initial velocity of their relative motion, then after emission of the neutrons, the velocities of the nuclei are

$$v'_1 \approx v_1 - i\frac{\sqrt{2mI_1}}{M_1}, \qquad v'_2 \approx v_2 - i\frac{\sqrt{2mI_2}}{M_2}, \tag{2}$$

and the velocities of the emitted neutrons

$$v_{n_1} \approx v_1 + i\sqrt{\frac{2I_1}{m}}, \qquad v_{n_2} \approx v_2 + i\sqrt{\frac{2I_2}{m}}. \tag{3}$$

If one of the nuclei (e.g., 2) first absorbs a neutron and then emits one (which in our notation corresponds to $t_3 > t_1 > t_2 > t_0$), then the velocity of the emitted neutron is equal to

$$v_{n_2} \approx v_2 + i\sqrt{\frac{2}{m}}\left[\sqrt{I'_2} - \frac{m}{M_2}\sqrt{I_1}\right] \approx v_2 + i\sqrt{\frac{2I'_2}{m}} \tag{4}$$

i.e., the exchange of t_1 and t_2 merely amounts to the substitution of I'_2 for I_2 (or, respectively I'_1 for I_1, if we are dealing with the emission of a neutron by nucleus 1). We observe that the distance ξ_0 between the nuclei at the instant t_0 of emission of the first neutron is connected with the energy E_0 by the relation

$$E_0 = \frac{Z_1 Z_2 e^2}{\xi_0} + \frac{\mu v_0^2}{2} = \frac{Z_1 Z_2 e^2}{\xi_0} + \frac{M_1}{\mu}\frac{M_1 v_1^2}{2}. \tag{5}$$

Inasmuch as the energy of the neutron $E_{n_1} = \frac{1}{2} m v_{n_1}^2$ is necessarily real, then either $v_1 = w - i\sqrt{(2I_1/m)}$, where w is a real magnitude, or v_1 is a purely imaginary magnitude, and $E_0 < (Z_1 Z_2 e^2/\xi_0)$, i.e., the emission of a neutron takes place inside the Coulomb potential barrier. One easily arrives at the same conclusions regarding v_2. Let us assume, to be definite in the following, that $t_3 > t_2 > t_1 > t_0$. Then for the expression Im $(S_I + S_{II})$ we find

$$\begin{aligned}\text{Im}\,(S_I+S_{II}) = \text{Im}\left\{\int_{\xi_0}^{\infty}\sqrt{2\mu\left(E_0 - \frac{Z_1 Z_2 e^2}{x}\right)}\,dx + \int_{\xi_1}^{\xi_0}\sqrt{2\mu_{01}\left(E_{01} - \frac{Z_1 Z_2 e^2}{x}\right)}\,dx\right.\\ \left. + \int_{\xi_2}^{\xi_1}\sqrt{2\mu_{12}\left(E_{12} - \frac{Z_1 Z_2 e^2}{x}\right)}\,dx + \int_{\xi_3}^{\xi_2}\sqrt{2\mu_{23}\left(E_{23} - \frac{Z_1 Z_2 e^2}{x}\right)}\,dx\right. \end{aligned} \tag{6}$$

$$+\int_{\infty}^{\xi_3}\sqrt{2\mu\left(E_3-\frac{Z_1Z_2e^2}{x}\right)}\,dx+\int_{x_{c_1}}^{x_{c_0}}(M_1+M_2-m)V_{01}\,dx$$
$$+\int_{x_{c_2}}^{x_{c_1}}(M_1+M_2-2m)V_{12}\,dx+\int_{x_{c_3}}^{x_{c_2}}(M_1+M_2-m)V_{23}\,dx \qquad (6)$$
$$+\int_{x_{n_1}}^{x_{n_1}}mv_{n_1}\,dx+\int_{x_{n_2}}^{x_{n_2}}mv_{n_2}\,dx\Big\}.$$

The first five terms in (6) characterize the relative motion of the interacting nuclei, the next three terms the motion of the centre of mass of these nuclei, and the last two the motion of the neutrons exchanged by the nuclei.

The general solution of the problem of the minimum of (6) is rather complicated. We shall therefore assume, as was done in ref.[3]), that E_0 is sufficiently large ($E_0 \gg Im/M$) and neglect terms of higher order in m/M.

Consider the value ξ lying inside the Coulomb potential barrier near its outer boundary, so that

$$\frac{Z_1Z_2e^2}{\xi_0}\approx E_0\left(1+c\frac{m}{\mu}\right) \qquad (7)$$

where $c\approx 1$.

It follows from (5) that

$$v_1=\frac{i\sqrt{2cmE_0}}{M_1}, \qquad (8)$$

$$v'_1=\frac{i\sqrt{2m}}{M_1}(\sqrt{cE_0}-\sqrt{I_1}), \qquad (9)$$

$$v_{n_1}=i\sqrt{\frac{2}{m}}\left(\frac{m}{M}\sqrt{cE_0}+\sqrt{I_1}\right)\approx i\sqrt{\frac{2I_1}{m}}. \qquad (10)$$

Similar relations can be obtained for v'_2 and v_{n_2}.

It is essential for what follows that only the two last terms in (6) which describe the motion of the neutrons do not include the factor m/M. The remaining terms contain this factor, and hence they are all negligible in comparison with the last two terms at ξ_0 close to $Z_1Z_2e^2/E_0$.

As in the problem of neutron transfer (ref.[3])), $\xi_0=Z_1Z_2e^2/E_0$ corresponds to the minimum of Im (S_I+S_{II}). At larger ξ_0 the expression (6) increases due to the last two terms; at smaller ξ_0 the first five terms rapidly increase due to Coulomb repulsion. It is easily seen that the paths traversed by the neutrons are equal to

$$l_{n_1}=|{}_0x_{n_1}-{}_2x_{n_1}|=|v_{n_1}(t_2-t_0)|=\xi_0-v_2(t_1-t_0)-v'_2(t_2-t_1), \qquad (11)$$
$$l_{n_2}=|{}_1x_{n_2}-{}_3x_{n_2}|=|v_{n_2}(t_3-t_1)|=\xi_0-v_2(t_1-t_0)-v'_1(t_3-t_0), \qquad (12)$$

so that with an accuracy up to terms of the order of m/M,

$$l_{n_1} \approx l_{n_2} \approx \xi_0 \approx \frac{Z_1 Z_2 e^2}{E_0}. \tag{13}$$

Hence, neutron exchange is most probable on the outer boundary of the Coulomb potential barrier, and

$$\mathrm{Im}\,(S_{\mathrm{I}}+S_{\mathrm{II}}) \approx i\sqrt{2m}\,\frac{Z_1 Z_2 e^2}{E_0}\,(\sqrt{I_1}+\sqrt{I_2}) \tag{14}$$

while the probability of direct neutron exchange interaction is proportional to

$$\phi = \exp\left\{-\frac{2\sqrt{2}Z_1 Z_2 e^2\sqrt{m}}{\hbar E_0}\,(\sqrt{I_1}+\sqrt{I_2})\right\}. \tag{15}$$

In the case when $I'_1 < I_1$ and $I'_2 < I_2$, the more probable process is that in which a neutron first attaches itself to nucleus 1 or 2 and only after this does the affected nucleus emit a neutron "in exchange."

In this case the final expression for ϕ should contain instead of $\sqrt{I_1}+\sqrt{I_2}$ the lesser of the two sums $\sqrt{I'_1}+\sqrt{I_2}$ or $\sqrt{I_1}+\sqrt{I'_2}$.

3. Supplementary Remarks

The relations derived above are similar, and in some parts even identical with those obtained by E. M. Lifshitz [3]) for the transfer of a neutron from one complex nucleus to another. Our final result — the expression for ϕ — differs from that given in ref.[3]) only in that it contains the sum of two terms $\sqrt{I_i}$ instead of one term, $\sqrt{I_1}$ or $\sqrt{I_2}$. Hence, in order to test the applicability of these relations and to estimate the effective values of the pre-exponential factors it is expedient to compare the formulas obtained in ref.[3]) with the large experimental material that has been recently accumulated on the neutron-transfer reactions for different nuclei bombarded with N^{14}-ions [4–9]).

For N^{14} we have $Z_1 = 7$, $I_1 = 10.55$ MeV, $E_0 = (14/14+M_2)E_{N_{lab}}$, so that

$$\frac{\alpha}{E_0} = \frac{2}{\hbar}\,\frac{Z_1 Z_2 e^2}{E_0}\,\sqrt{mI} = \frac{(14+M_2)\ 14.3\ Z_2}{M_2 E_{N_{lab}}(\mathrm{MeV})}.$$

The experimental data [5–8]) on the dependence of the neutron-transfer cross sections on the N^{14}-ions are represented in fig. 1 by the values of the logarithms of the pre-exponential factor

$$\lg k = \lg \sigma(\mathrm{mb}) + \frac{\alpha}{2.3\,E_{N_{lab}}}$$

for different $E_{N_{lab}}$. As appears from the figure, for at least six nuclei (Be^9, Na^{23}, Al^{27} and three isotopes of magnesium) the sum $\lg\sigma+(\alpha/2.3\,E_{N_{lab}})$ is actually constant, which confirms the qualitative applicability of the relations under discussion.

The data on the neutron-transfer from N^{14} to B^{10} and N^{14} do not fit the relation deduced in ref.[3]); the cross sections of these reactions increase too slowly with the energy (α_{exp} is smaller than the value given above) and the magnitudes of $\lg \sigma + (\alpha/2.3\, E_{N_{lab}})$ decrease with increase in the energy of the nitrogen ions.

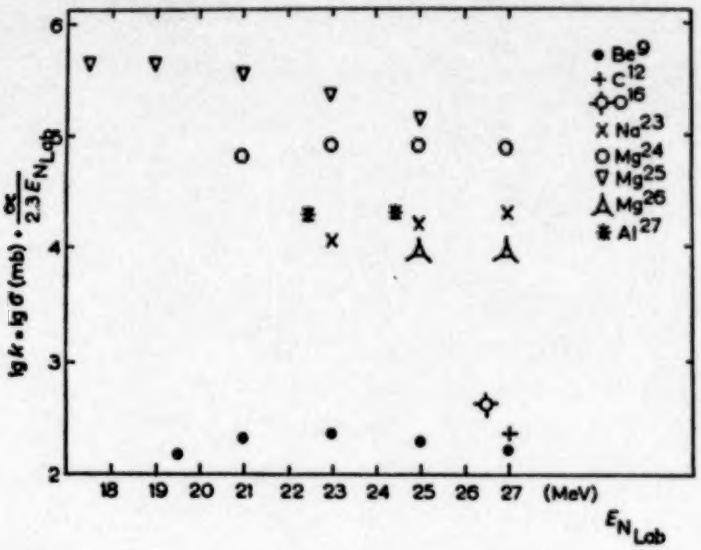

Fig. 1

However, in these cases, too, the discrepancy between the experimental data and the ordinary formula for the probability of penetration through the Coulomb barrier is far stronger.

The values of the pre-exponential factors derived from fig. 1 (0.2—0.4 barn for Be, C, O and 10—100 barns for Na, Mg, Al) are, of course, too high, inasmuch as in the relations given above the nuclei were regarded as points.

We shall not go into further detail regarding the influence of the finite dimensions of the nucleus on the exact calculations, but merely observe that the region of rapid growth of the neutron-transfer cross section for N^{14} nuclei with the energy of the latter, as observed in the experiments, extends to energies slightly exceeding the values usually accepted for the Coulomb potential barrier. We are not concerned here, however, with the exact contribution of the barrier to the effective values of the pre-exponential factors indicated above; what is of importance is that the true values of the pre-exponential factors are in order of magnitude close to those expected for the direct neutron exchange interaction. In the latter case one can expect only a comparatively slight decrease of the pre-exponential factor for the formation of each of the final states with different spins due to the possibility of many such states. Hence, a comparison of the values of geometrical and neutron-transfer cross sections, as well as of ϕ for neutron exchange and neutron-transfer, allows us to conclude that the cross sections of direct neutron

exchange interaction of complex nuclei near the barrier energy may attain magnitudes of 10^{-30}—10^{-29} cm^2. Hence, such interaction may be observed by looking for the subsequent γ-radiation (for example, for coincidences of γ-rays emitted by both nuclei in final states) and in a number of cases also by possibly detecting yet unknown isomer transitions, viz. M1 transitions of comparatively long period and low energy.

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UNRENORMALIZABLE INTERACTIONS AND THE STRUCTURE OF ELEMENTARY PARTICLES

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Abstract: A possible relation of the unrenormalizable interactions to the internal structure of elementary particles is approached with the help of a simple model. A non-local, charge-independent, and CP (combined) invariant interaction is examined and a simple connection is found to emerge between the problems of parity violation in weak interactions and the divergence of self-energy. The connection involves a supposed universal fundamental length.

In 1936 Heisenberg grouped interactions into two classes according to the dimensionality of the coupling constant involved [1]: the first class including those interactions with dimensionless coupling constants, and the second class all others. It was later discovered that first class interactions are renormalizable, while second class ones are not [2]), a circumstance which cast doubt on the credentials of the latter type of interaction.

The accumulating weight of experimental data is, however, beginning to demand the reinstatement of second class interactions. The investigations, in particular, by Umezawa *et al.* of pion asymmetry in baryon decay support this conclusion [3]). An even more thoroughgoing revision of the techniques of field theory is indicated by the recent Stanford experiments on electron-nucleon scattering [4]). These seem to suggest that the interaction is definitely a non-local one †. It would appear that the internal structure of the "elementary" particles cannot be dismissed.

It is our view that the apparent finite size of the elementary particles is closely related to the problems of parity non-conservation and diverging self-energy. The connecting link is provided by the second class interaction. To give some credence to this prognostication we analyse a simple model.

Consider the hypothetical CP-invariant, charge-independent and non-local interaction between a fermion field, ψ, of isospin $\frac{1}{2}$ and a boson field, $\boldsymbol{\phi}$, of isospin 1 given by the interaction lagrangian ††

$$\mathscr{L}(x) = f\int\mathrm{d}^4\xi\bar{\psi}(\xi)\eta(\xi-x)\tau\psi(\xi)\cdot\boldsymbol{\phi}(x). \qquad (1)$$

Here f is a dimensionless coupling constant and $\eta(\xi-x)$ is a (proper)

† In this connection we thank Professor W. Heitler for a stimulating discussion.
†† This type of interaction was discussed by Z. Koba [5]).

Lorentz invariant matrix. More complicated situations could be discussed but this model is sufficient for illustrative purposes.

The expression (1) is manifestly charge-independent. A certain simplification can be achieved by taking advantage of the CP-invariance requirement. We suppose, for definiteness, the following transformation properties under space reflection:

$$\psi \to \gamma_4 \psi \quad \text{and} \quad \phi \to -\phi \tag{2}$$

and, under charge conjugation,

$$\psi \to C\bar{\psi}^{\mathrm{T}} \quad \text{and} \quad \phi \to \phi^{\mathrm{c}} \tag{3}$$

where C is the usual charge conjugation matrix [6]) and ϕ^{c} is to be determined in accordance with the invariance requirement. The possibilities become more apparent when the invariant $\eta(x)$ is expanded in terms of the 16-member γ-algebra

$$\eta(x) = \eta_0(x)+i\gamma_5\eta_1(x)+il\gamma_5\gamma_\mu\partial_\mu\eta_2(x)+il\gamma_\mu\partial_\mu\eta_3(x). \tag{4}$$

The Lorentz invariant functions $\eta_i(x)$ are real-valued and of dimension (length)$^{-4}$. This is the most general η consistent with a hermitian Lorentz invariant interaction. The parameter l which appears explicitly in (4) is assumed to be contained implicitly in the functions $\eta_i(x)$, and in addition to be the *only* parameter so contained.

The expansion (4) can now be substituted in (1), and two of the terms integrated by parts. The resulting expression can then be subjected to the CP transformation with the help of (2) and (3). Two CP-invariant alternatives are, in fact, found to emerge:

(A) $$\phi_1^{\mathrm{c}} = -\phi_1, \qquad \phi_2^{\mathrm{c}} = \phi_2, \qquad \phi_3^{\mathrm{c}} = -\phi_3 \tag{5}$$

together with

$$\mathscr{L}(x) = f\int \mathrm{d}^4\xi\bar{\psi}(\xi)\tau\psi(\xi)\eta_0(\xi-x)\cdot\phi(x); \tag{5'}$$

or

(B) $$\phi_1^{\mathrm{c}} = \phi_1, \qquad \phi_2^{\mathrm{c}} = -\phi_2, \qquad \phi_3^{\mathrm{c}} = \phi_3 \tag{6}$$

together with

$$\begin{aligned}\mathscr{L}(x) = {} & if\int \mathrm{d}^4\xi\bar{\psi}(\xi)\gamma_5\tau\psi(\xi)\eta_1(\xi-x)\cdot\phi(x)\\ & +ilf\int \mathrm{d}^4\xi\bar{\psi}(\xi)\gamma_5\gamma_\mu\tau\psi(\xi)\eta_3(\xi-x)\cdot\partial_\mu\phi(x)\\ & +ilf\int \mathrm{d}^4\xi\bar{\psi}(\xi)\gamma_\mu\tau\psi(\xi)\eta_3(\xi-x)\cdot\partial_\mu\phi(x).\end{aligned} \tag{6'}$$

(We have used here ${\tau_1}^{\mathrm{T}} = \tau_1, \quad {\tau_2}^{\mathrm{T}} = -\tau_2, \quad {\tau_3}^{\mathrm{T}} = \tau_3$.)

We shall suppose that the constant l has some universal significance and is related specifically to the finiteness of elementary particles. In particular, we suppose that the interactions become local (and first class) when l

vanishes. This is the only possibility consistent with our assumptions on the constitution of the $\eta_i(x)$. Restricting our attention to the alternative (B) ((A) implies parity violation even at low energies, and is therefore unrealistic) we can write the interaction (6′) in the form

$$\begin{aligned}\mathscr{L}(x) &= if\bar{\psi}(x)\gamma_5\boldsymbol{\tau}\psi(x)\cdot\boldsymbol{\phi}(x)\\ &\quad +il f\bar{\psi}(x)(\alpha+\beta\gamma_5)\gamma_\mu\boldsymbol{\tau}\psi(x)\cdot\partial_\mu\boldsymbol{\phi}(x)+\mathrm{O}(l^2),\end{aligned} \tag{7}$$

where α and β are constants determined by η_2 and η_3. The dominant term for low energies evidently conserves parity. The situation for high energies (small distances), however, is radically different: the second class and parity non-conserving terms become important. By choosing the form factor η properly it is possible to have a finite self-energy for the fermion. If one uses only the first class term (i.e. lets $l = 0$), or, for that matter, the first several terms in calculating the self-energy, the result will be divergent. One has neglected interactions which are important in small regions — within the fermion.

This model is, of course, rather arbitrary, but it does exhibit clearly the fundamental connections, which were suggested above, between the diverging self-energy, parity violation in weak interactions, and the apparently finite size of elementary particles. Underlying these considerations is the concept of a fundamental length which we believe must form the centrepiece of a correct theory. It is perhaps not unreasonable to suppose that any realistic interaction (e.g. pion-baryon) would embody much the same features.

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